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A SIMPLIFICATION OF D'ALARCAO'S IDEMPOTENT SEPARATING EXTENSIONS OF INVERSE SEMIGROUPS

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ABSTRACT, In [2] D'Alarcao states necessary and sufficient conditions for the attainment of an idempotent-separating extension of an inverse semigroup. To do this D'Alarcao needed essentially three mappings satisfying thirteen conditions. In this paper we show that one can achieve the same results with two mappings satisfying eight conditions.

KEY WORDS AND PHRASES. Inverse semigroup and idempotent-separating extension.

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1. THE CONSTRUCTION.

Unless otherwise mentioned, the terminology and notation of Clifford and Preston [1] shall be assumed. The reader is referred to [2] for all other undefined terms.

LEMMA 1. If A and B are inverse semigroups and f is a homomorphism of A onto E(B) which is one-to-one on E(A), then

(i) $af = (a^{-1}a)f$ for all $a \in A$;

(ii) $a^{-1}ga = ga^{-1}a$ for all $a \in A$ and $g \in E(A)$; and

(iii) $ga = ag$ for all $a \in A$ and $g \in E(A)$.

PROOF. (i) Since $af \in E(B)$, by Lallement's result [3], there exists $e \in E(A)$ such that $af = ef$. Consequently, $ef = e^{-1}f = (ef)^{-1} = (af)^{-1} = a^{-1}f$, and thus $af = ef = (a^{-1}f)(af) = (a^{-1}a)f$.

(ii) By part (i), for each $g \in E(A)$ and $a \in A$, $((ga)^{-1}ga)f = (ga)f = gf \cdot af = gf \cdot (a^{-1}a)f = (ga^{-1}a)f$. Since f is one-to-one on $E(A)$, we have $a^{-1}ga = (ga)^{-1}ga = ga^{-1}a$.

(iii) By part (ii), $ga = a(a^{-1}ga) = a(ga^{-1}a) = ag$. This completes the proof of the lemma.

Let A and B be inverse semigroups and let f be a homomorphism of A onto $E(B)$ which is one-to-one on $E(A)$. (The existence of such an f implies that A is a semilattice of groups; see e.g. [4].) Suppose there is a mapping w of $B \times B$ into A (denoted by $(b,c)w = b^c$) and that for each $b \in B$ there is a mapping U_b of A into A (denoted by $aU_b = a^b$) such that:

(P1) For each $b, c \in B$, $b^c \in (c^{-1}b^{-1}bc)f^{-1}$.

(P2) For each $b \in B$, U_b is a homomorphism.

(P3) If $b, c \in B$ and $a \in (b^{-1}b)f^{-1}$, then $a^c \in (c^{-1}b^{-1}bc)f^{-1}$.

(P4) $(bc)^d(b^c)^d = b^{cd}c^d$ for all $b, c, d \in B$.

(P5) If $b \in E(B)$ and $a \in A$, then $a^b = ea$ where $\{e\} = E(A) \cap bf^{-1}$.

(P6) For each $b \in B$, $(bb^{-1})^b = e$ where $\{e\} = E(A) \cap (b^{-1}b)f^{-1}$.

(P7) $b^c(a^b)^c = a^{bc}b^c$ for all $a \in A$ and $b, c \in B$.

(P8) For each $b, c \in E(B)$, $b^c = eg$ where $\{e\} = E(A) \cap bf^{-1}$ and $\{g\} = E(A) \cap cf^{-1}$.

Let $S^* = \{(b,a): b \in B \text{ and } a \in (b^{-1}b)f\}$. Let equality in S^* be defined in the usual manner and let multiplication be defined by $(b,a)(c,p) = (bc, b^c a^c p)$.

Moreover, let $\gamma : S^* \to B$ be defined by $(b,a)\gamma = b$.

THEOREM 1. Under the above conditions, (S^*,γ) is an idempotent-separating extension of A by B. Moreover, $\gamma|_A = f$.

PROOF. Note that since fof^{-1} is idempotent separating, $fof^{-1} \subseteq \mathcal{H}$. Consequently, if $e \in E(A)$ and $e, a \in (b^{-1}b)f^{-1}$, then $ea = a = ae$.

As is shown in [2], closure in S^* follows from (P1) and (p3) and, associativity follows from (P2), (P4) and (P7).

Our proof that S^* is regular is the same as that found in [2] but is more informative. Let $(b,a) \in S^*$ and let $x = \left(a^{b^{-1}}\right)^{-1}\left(b^{b^{-1}}\right)^{-1}$. By (P1) and (P3), $\left(\left(a^{b^{-1}}\right)^{-1}\right)f = bb^{-1} = \left(\left(b^{b^{-1}}\right)^{-1}\right)f$. Consequently $(b^{-1},x) \in S^*$ since $xf = \left(\left(a^{b^{-1}}\right)^{-1}\left(b^{b^{-1}}\right)^{-1}\right)f = bb^{-1}$. Also, by (P1) and (P3), $\left(b^{b^{-1}}a^{b^{-1}}x\right)^b f = b^{-1}b$. Since $\left(b^{b^{-1}}a^{b^{-1}}x\right)^b \in E(A) \cap (b^{-1}b)f^{-1}$ we have $\left(b^{b^{-1}}a^{b^{-1}}x\right)^b a = a$. It now follows from (P6) that $(b,a)(b^{-1},x)(b,a) = \left(b,(bb^{-1})^b\left(b^{b^{-1}}a^{b^{-1}}x\right)^b a\right) = (b,ea) = (b,a)$ since $\{e\} = E(A) \cap (b^{-1}b)f^{-1}$. Consequently, S^* is regular.

As shown in [2], it follows from (P5) and (P6) that if $(g,e) \in E(S^*)$ then $g \in E(B)$ and $\{e\} = E(A) \cap gf^{-1}$. Consider the subset $A^* = \{(b,a) : b \in E(B)\} \subseteq S^*$ which contains $E(S^*)$. Let $\delta : A^* \to A$ by $(b,a) \to a$. To show δ is a homomorphism, let $(b,a), (c,p) \in A^*$. Then $a = ea$ and $p = gp$ where $\{e\} = E(A) \cap bf^{-1}$ and $\{g\} = E(A) \cap cf^{-1}$. By (P8), (P5) and lemma 1iii, $(b,a)(c,P) = (bc, b^c a^c p) =$ $= (bc, eg \cdot ga \cdot p) = (bc, ea \cdot gp) = (bc, ap)$. Thus δ is a homomorphism which obviously has A for its range. It follows from the definitions of S^* and δ that δ is one-to-one. Consequently the idempotents of S^* commute since A^* is isomorphic to the inverse semigroup A. We now have that S^* is an inverse semigroup.

It is obvious that γ is a homomorphism of S^* onto B and $\{E(B)\}\gamma^{-1} = A^*$. If (b,a), $(c,p) \in E(A^*)$ such that $(b,a)\gamma = (c,p)\gamma$, then $b = c$ and $af = b = c = pf$. Since $a, p \in E(A)$, $a = p$. Consequently γ is one-to-one on $E(A^*)$. It is obvious that $\gamma|_{A^*} = f$. Since $A^* = A$, this completes the proof of the theorem.

2. THE STRUCTURE THEOREM.

Since our two mappings and eight properties are the same as, but fewer than, those used by D'Alarcao, the following theorem has been proven by D'Alarcao ([2], Theorem 2).

THEOREM 2. Let A and B be inverse semigroups and let (S,f) be an idempotent-separating extension of A by B. Then for each $b \in B$ there exists a mapping U_b of A into A (denoted by $aU_b = a^b$) and there exists a mapping w of $B \times B$ into A satisfying (P1) through (P8). Moreover, S is isomorphic to S^*.

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