A NOTE ON RIESZ ELEMENTS IN C^* -ALGEBRAS

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ABSTRACT. It is known that every Riesz operator R on a Hilbert space can be written $R = Q + C$, where C is compact and both Q and $CQ - QC$ are quasinilpotent. This result is extended to a general C^* -algebra setting.

1. INTRODUCTION.

In [3], Smyth develops a Riesz theory for elements in a Banach algebra with respect to an ideal of algebraic elements. In [1], Chui, Smith and Ward show that every Riesz operator on a Hilbert space is decomposable into $R = Q + C$, where C is compact and both Q and $CQ - QC$ are quasinilpotent. In this paper we use Smyth's work to show that the analogous result holds in an arbitrary C^* -algebra.

2. DEFINITIONS AND NOTATION.

Let A be a C^* -algebra, and let F be a two-sided ideal of algebraic elements

of A . An element $T \in A$ is a Riesz element if its coset $T + \bar{F}$ in $A/\bar{F}$ has spectral radius 0. A point $\lambda \in \sigma(T)$ is a finite pole of T if it is isolated in $\sigma(T)$ and the corresponding spectral projection lies in F . Let $E\sigma(T) = \{\lambda \in \sigma(T) : \lambda \text{ is not a finite pole of } T\}$. Smyth has shown that T is a Riesz element if and only if $E\sigma(T) \subseteq \{0\}$, [3, Thm. 5.3]. Smyth also showed that if T is a Riesz element, then $T = Q + U$, where Q is quasinilpotent and $U \in \bar{F}$. [3, Thm. 6.9]. This is a generalization of West's result [4, Thm. 7.5]. We now extend the result of Chui, Smith and Ward [1, Thm. 1] by showing that $UQ - QU$ is quasinilpotent, where $T = Q + U$ is the Smyth decomposition.

3. OUTLINE OF SMYTH'S CONSTRUCTION.

Let T be a Riesz element, and label the elements of $\sigma(T) \setminus E\sigma(T)$ by λ_n , $n = 1, 2, \dots$, in such a way that $|\lambda_n| \geq |\lambda_{n+1}|$, $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$. Each λ_n is a finite pole, so each spectral projection P_n is in F . Let $S_n = P_1 + \dots + P_n$, then find a self-adjoint projection Q_n satisfying $S_n Q_n = Q_n$ and $Q_n S_n = S_n$. Let $V_n = Q_n - Q_{n-1}$, and define $U = \sum \lambda_k V_k$. U is clearly in $\bar{F}$ and $Q = T - U$ is shown to be quasinilpotent.

4. THEOREM 1 $UQ - QU$ is quasinilpotent.

PROOF. For any $S \in A$, let $\tilde{S}$ denote the left regular representation of S . Then by Lemma 6.6 in Smyth [3], we have that $Q_n A$ is an invariant subspace of $\tilde{Q}$. Since $Q_n = Q_n Q_n$, we have $Q_n \in Q_n A$. Hence $\tilde{Q}(Q_n) \in Q_n A$, say $\tilde{Q}(Q_n) = Q_n S$ for some $S \in A$. That is, $QQ_n = Q_n S$. Now let $v \in \text{range } Q_n$, say $v = Q_n x$. Then $Qv = QQ_n x = Q_n Sx$ belongs to $\text{range } Q_n$. Hence we see that $\text{range } Q_n$ is an invariant subspace of Q . It follows that Q has an operator matrix representation of the form

$$Q = \begin{array}{cccc|c} A_{11} & A_{12} & A_{13} & \cdots & \\ \circ & A_{22} & A_{23} & \cdots & * \\ \circ & \circ & A_{33} & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \\ \hline & & \circ & & * \end{array}$$

where $A_{ij} = V_i Q V_j$. With respect to this blocking, we have

$$U = \begin{array}{cccc|c} \lambda_1 I_1 & \circ & \circ & \cdots & \\ \circ & \lambda_2 I_2 & \circ & \cdots & \\ \circ & \circ & \lambda_3 I_3 & \cdots & \circ \\ \vdots & \vdots & \vdots & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \\ \hline & & \circ & & \circ \end{array}$$

$$\begin{array}{l} \text{Hence} \\ \begin{array}{cccc|c} \circ & (\lambda_1 - \lambda_2) A_{12} & \cdots & (\lambda_1 - \lambda_n) A_{1n} & \cdots \\ \circ & \circ & & & \\ \vdots & \vdots & \ddots & & \\ \vdots & \vdots & \vdots & (\lambda_{n-1} - \lambda_n) A_{n-1,n} & \cdots \\ \hline & & \circ & & \end{array} \\ \text{UQ - QU} = \begin{array}{cccc|c} & & & & \\ & & & & \\ & & & & \\ & & & & \\ \hline & & \circ & & \circ \end{array} \end{array}$$

Now let P be the orthogonal projection onto $\bigcup_n \text{range } Q_n$, and let

$A_n = (P - Q_n)(UQ - QU)(P - Q_n)$. It is easy to see that $\|A_n\| \leq \lambda_n \|Q - \text{diag. } Q\| \rightarrow 0$ as $n \rightarrow \infty$. Hence $UQ - QU - A_n$ converges in the uniform norm to $UQ - QU$ as $n \rightarrow \infty$.

But $UQ - QU - A_n$ has the form

N	*	*
0	0	*
0	0	0

where N is nilpotent. It follows that $UQ - QU - A_n$ has no non-zero eigenvalues.

Thm. 3.1, p. 14 of [2] can now be easily modified to show that $UQ - QU$ has no non-zero eigenvalues. Since $UQ - QU$ belongs to $\overline{F}$, this means $\sigma(UQ - QU) \subseteq \{0\}$, i.e., $UQ - QU$ is quasinilpotent.

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