ABSTRACT. The purpose of this note is to establish a connection between the notion of (n-2)-tightness in the sense of N.H. Kuiper and T.F. Banchoff and the total absolute curvature of compact submanifolds-with-boundary of even dimension in Euclidean space. The argument used is a certain geometric inequality similar to that of S.S. Chern and R.K. Lashof where equality characterizes (n-2)-tightness.

KEY WORDS AND PHRASES. tight manifolds, total absolute curvature.

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1. **INTRODUCTION.**

Let $M$ be a compact $n$-dimensional smooth manifold with or without boundary — where the boundary is assumed to be smooth — and let

$$f : M \longrightarrow \mathbb{E}^{n+k}$$

be a smooth immersion of $M$ into the $(n+k)$-dimensional euclidean space. This leads to the notion of total absolute curvature

$$TA(f) = \frac{1}{c_{n+k-1}^M} \int_N |K| * 1$$

where $K$ denotes the Lipschitz-Killing curvature of $f$ in each normal direction, $N$ the unit normal bundle (with only the 'outer' normals at points of $\partial M$), and $c_m$ denotes the volume of the unit sphere $S^m \subseteq \mathbb{E}^{m+1}$. For detailed definitions, in particular in the case of manifolds with boundary, see [5] or [6].

Let us state the following equation ([6], 2.2)

$$TA(f) = TA(f|_{M \setminus \partial M}) + \frac{1}{2}TA(f|_{\partial M})$$

(1.1)

The famous result of S.S. Chern and R.F. Lashof gives a connection between total absolute curvature and the number of critical points of so-called height functions

$$zf : M \longrightarrow \mathbb{R}$$

defined by $(zf)(p) = \langle z, f(p) \rangle$, $z \in S^{n+k-1}$

Extending this result to the case of manifolds with boundary we can write

$$TA(f) = \frac{1}{c_{n+k-1}^M} \int_{S^{n+k-1}} \sum_i (\mu_i(zf) + \mu^+_i(zf)) * 1$$

(1.2)

where $\mu_i(zf)$ denotes the number of critical points of $zf$ of index $i$ in $M \setminus \partial M$, and $\mu^+_i(zf)$ denotes the number of (+)-critical points of $zf$ of index $i$ in $\partial M$. Here a point $p \in \partial M$ is called (+)-critical if $p$ is critical...
for $z f \mid_{p \in M}$ and $\text{grad}_p f$ is a nonvanishing inner vector on $M$ (for details, see [2], [4] or [6]).

The $i$-th curvature $\tau_i$ introduced by N.H. Kuiper (cf. [7]) can be expressed by

$$\tau_i(f) = \frac{1}{c_{n+k-1}} \int_{S^{n+k-1}} (\mu_i(zf) + \mu_i(zf)^*1) \quad (cf. [6], lemma 4.2 or [9], lemma 3.1).$$

So we get

$$\text{TA}(f) = \sum_i \tau_i(f),$$

The Morse-relations give the following connections between the curvatures and some topological invariants of $M$:

$$\tau_i(f) \geq b_i(M)$$
$$\text{TA}(f) \geq b(m) := \sum_i b_i(M)$$
$$\sum_i (-1)^i \tau_i(f) = \chi(M) = \sum_i (-1)^i b_i(M)$$

where $b_i(M)$ denotes the $i$-th Betti-number of homology with coefficients in a suitable field. (cf. [7]).

$f$ is called $k$-tight if for all $k' \leq k$ and for almost all $z \in S^{n+k-1}$ and all real numbers $c$ the inclusion map

$$j: (zf) := \{p \in M/ (zf)(p) \leq c\} \longrightarrow M$$

induces a monomorphism in the $k'$-th homology:

$$H_{k'}(j) : H_{k'}((zf)_c) \longrightarrow H_{k'}(M)$$

As usual we write shortly 'tight' instead of 'n-tight'.

Then the results of N.H. Kuiper show

$$\text{TA}(f) = b(M) \quad \text{if and only if} \quad f \text{ is tight},$$
\( \tau_k(f) = b_k(M) \) if and only if \( H_k(j) \) and \( H_{k-1}(j) \) are monomorphisms for almost all \( z, c \) (cf. [7]).

Results on tightness are collected in the survey article [10] by T.J. Willmore, for results on \( k \)-tightness we refer in addition to the notes [1] by T. Banchoff and [9] by L. Rodríguez, who has shown that in some sense \((n-2)\)-tightness is closely related to convexity.

2. RESULTS

As mentioned above there is a relation between tightness on one hand and total absolute curvature and the sum of the Betti-numbers on the other hand. The following results give certain connections between \((n-2)\)-tightness on one hand and usual curvature terms and the sum of the Betti-numbers on the other hand. Note that in case \( \partial M = \emptyset \) by duality arguments tightness is equivalent to \( k \)-tightness for \( k = \frac{n}{2} - 1 \) if \( n \) is even and for \( k = \frac{n-1}{2} \) if \( n \) is odd. But in case \( \partial M \neq \emptyset \) there are examples of \((n-2)\)-tight immersions which are not tight (for example: consider the round hemi-sphere).

THEOREM A. Let \( M^n \) be an even-dimensional manifold with non-void boundary and \( f : M \to \mathbb{R}^{n+k} \) be an immersion. Let \( N_0 \) be the unit normal bundle of \( f \) and denote by \( N_* \subset N_0 \) the open set of unit normals where the second fundamental form of \( f \) is positive or negative definite.

Then there holds the following inequality

\[
\frac{1}{2} \text{TA}(f) \bigg|_{\partial M} + \frac{1}{c_{n+k-1}} \int_{N_0 \setminus N_*} |K| \geq b(m) \tag{2.1}
\]

where equality characterizes \((n-2)\)-tightness of \( f \).

In case of hypersurfaces \((k = 1)\) (2.1) becomes

\[
\frac{1}{2} \text{TA}(f) \bigg|_{\partial M} + \text{TA}(f) \bigg|_{M \setminus M_{\text{rel}}} \geq b(M) \tag{2.2}
\]
where \( M_\ast \) denotes the set of points in \( M \cap M \) with positive or negative definite second fundamental form.

In case \( n = 2 \) \( M_\ast \) is just the set of points with positive Gaussian curvature, so we get

**COROLLARY A 1.** Assume \( n = 2 \) and \( k = 1 \). Then there holds the following inequality

\[
\frac{1}{2\pi} \int_{K < 0} |K| \, do + \frac{1}{2\pi} \int_{\partial M} |\kappa| \, ds > b(M) \geq 2 - \chi(M)
\]

where equality characterizes 0-tightness of \( f \). Here \( |\kappa| \) denotes the usual curvature of \( f|_{\partial M} \) considered as a space curve. For part of this result see [8], Prop. 9.

**COROLLARY A 2.** Assume \( b(M\cap M) = 2 b(M) \). Then \((n-2)\)-tightness of \( f \) implies that \( f|_{\partial M} \) is tight and that the second fundamental form of \( f \) has either non-maximal rank or is positive or negative definite.

This is shown in [9], Prop. 5.2 under the assumption that \( M^n \) can be embedded in \( E^n \). This condition implies \( b(M\cap M) = 2 b(M) \) by Alexander duality.

Under the additional assumption that \( M \) consists of a certain number of \((n-1)\)-spheres L. Rodriguez has shown that \((n-1)\)-tightness is equivalent to convexity (cf. [9], Theorem 2). This is not true in general, (See Corollary B 2 below).

**THEOREM B.** Let \( n \) be even and \( f : M^n \to E^{n+1} \) be \((n-2)\)-tight (if \( \partial M \neq \phi \)) or tight (if \( \partial M = \phi \)), and let \( \tilde{M} \subseteq M\cap M \) be a compact submanifold of dimension \( n \) which is contained in some coordinate neighborhood in \( M \). As above \( M_\ast \) denotes the set of points in \( M\cap M \) with positive or negative definite second fundamental form. Then there holds the following inequality
where equality characterizes \((n-2)\)-tightness of \(f|_{M \setminus \tilde{M}}\).

**Remark.** If \(\tilde{M}\) contains only points of vanishing curvature or definite second fundamental form, then \(\tilde{M} \cap M = \emptyset\) and (2.4) reduces to the inequality of S.S. Chern and R.K. Lashof for \(\tilde{M}\), otherwise (2.4) is sharper and reflects the additional condition that \(M\) lies inside of some given \(M\). For example in case \(n = 2\) and \(\tilde{M}\) being a disk we get

\[
\int_{\tilde{M}} \sqrt{\chi} \, ds > 2\pi + \int_{M \cap \{K < 0\}} |K| \, ds \tag{2.5}
\]

**Corollary B 1.** Let \(f\) be as in Theorem B and assume moreover that there is an open region \(U \subseteq M\) which is embedded by \(f\) in a hyperplane of \(E^{n+1}\) which implies \(K|_{U} = 0\). Let \(M^n\) be an embedded compact submanifold of \(E^n\) and assume by changing the scale \(\tilde{M} \subseteq f(U)\).

Then \(f^{-1}(M \setminus \tilde{M})\) is \((n-2)\)-tight if and only if \(\tilde{M}\) is tightly embedded in \(E^n\).

Note that for \(\tilde{M} \subseteq E^n\) tightness of \(\tilde{M}\) and tightness of \(M\) are equivalent: this can be obtained easily using the equations \(TA(\tilde{M}) = \frac{1}{2}TA(M)\) and \(b(\tilde{M}) = \frac{1}{2}b(M)\).

Roughly spoken Corollary B 1 says: \((n-2)\)-tight minus tight gives \((n-2)\)-tight. In particular we get the following

**Corollary B 2.** In each even dimension there exist \((n-2)\)-tight hypersurfaces which are not tight and not convex in the sense of [9], in particular where \(f(\tilde{M})\) is not contained in the boundary of the convex hull of \(f(M)\).

3. **Proofs.**

In all proofs the immersion \(f\) is fixed and so we may write \(TA(\tilde{M})\) instead.
of $TA(\mathcal{f}|_{\mathcal{M}})$ and so on.

**PROOF OF THEOREM A.**

From

$$TA(M) = \sum_{i=1}^{\infty} \tau_i(M)$$

and

$$\chi(M) = \sum_{i=1}^{\infty} (-1)^i \tau_i(M)$$

we get

$$TA(M) + \chi(M) = 2\sum_{i=1}^{\infty} \tau_i(M)$$

On the other hand by definition $\tau_n(M)$ is the average of the number of critical points of $zf$ of index $n$ which are precisely the strict local maxima in $M \setminus \mathcal{M}$. But a point is a strict local extremum of some height function $zf$ if and only if the second fundamental form in the direction of $z$ is positive or negative definite. Hence we get

$$2 \tau_n(M) = \frac{1}{c_{n+k-1}} \int_{N_*} |K| \ast 1$$

leading to

$$TA(M) - \frac{1}{c_{n+k-1}} \int_{N_*} |K| \ast 1$$

$$= 2 (\tau_0(M) + \tau_2(M) + \ldots + \tau_{n-2}(M)) - \chi(M)$$

$$\geq 2 (b_0(M) + b_2(M) + \ldots + b_{n-2}(M)) - \chi(M)$$

$$= b(M),$$

where we have used the assumption that $n$ is even and $\mathcal{M} \neq \emptyset$ which implies $b_n(M) = 0$.

The case of equality is equivalent to the following equations:
\[ \tau_0(M) = b_0(M), \tau_2(M) = b_2(M), \ldots, \tau_{n-2}(M) = b_{n-2}(M) \quad (2.6) \]

But the equality \( \tau_1(M) = b_1(M) \) is equivalent to injectivity of \( H_i(j) \) and \( H_{i-1}(j) \) for all inclusions \( j : (zf)_c + M \), so (2.6) is equivalent to (n-2)-tightness of \( f \).

The assertion of the theorem then follows from the inequality above using the equation (1.1)

\[ TA(M) = TA(M \setminus \partial M) + \frac{1}{2} TA(\partial M) \]

PROOF of Corollary A 2. By theorem A (n-2)-tightness of \( f \) implies

\[ b(M) = \frac{1}{2} TA(\partial M) + \frac{1}{c_{n+k-1}} \int_{N_0 \setminus N_*} |K| * 1 \]

\[ \geq \frac{1}{2} TA(\partial M) \geq \frac{1}{2} b(\partial M) = b(M) \]

which implies tightness of \( f \bigg|_{\partial M} \) and moreover the vanishing of the integral of \( |K| \) over \( N_0 \setminus N_* \), hence \( K = 0 \) on \( N_0 \setminus N_* \).

PROOF of Theorem B. By assumption and by theorem A we have

\[ TA(M \setminus M_\phi \setminus \partial M) + \frac{1}{2} TA(\partial M) = b(M), \text{ if } \partial M \neq \phi, \quad (2.7) \]

or

\[ TA(M) = b(M), \text{ if } \partial M = \phi \]

which last equality is equivalent to

\[ TA(M \setminus M_\phi) = b(M) - 2 \quad (2.8) \]

For \( f \bigg|_{M \setminus (M \setminus \partial M)} \) theorem A yields

\[ TA(M \setminus M_\phi \setminus \partial M \setminus \partial \tilde{M}) + \frac{1}{2} TA(\partial \tilde{M}) + \frac{1}{2} TA(\partial \tilde{M}) \geq b(M \setminus \tilde{M}) \quad (2.9) \]

where equality characterizes (n-2)-tightness of \( f \bigg|_{M \setminus (M \setminus \partial \tilde{M})} \).

Subtracting (2.9) from (2.7) or (2.8) respectively we get
respectively.

Now the assertion follows directly from the following lemma

**Lemma.** Let $M, \tilde{M}$ be $n$-dimensional compact connected manifolds with $\tilde{M} \subseteq M \setminus \partial M$ and assume that $\tilde{M}$ is contained in some coordinate neighborhood of $M$.

Then

$$b(M \setminus \tilde{M}) - b(M) = \frac{1}{2} b(\partial \tilde{M})$$

if $\partial M \neq \emptyset$, or

$$b(M \setminus \tilde{M}) - b(M) = \frac{1}{2} b(\partial \tilde{M}) - 2$$

if $\partial M = \emptyset$.

**Proof.** Let $\tilde{B}$ be an open coordinate neighborhood in $M$ such that $\tilde{B}$ is topologically a closed $n$-ball. We can assume $\tilde{M} \subseteq \tilde{B} \subseteq \overline{B} \subseteq M \setminus \partial M$. To compute the Betti-numbers of $M \setminus \tilde{M}$ in terms of that of $M$ and $\tilde{M}$ we apply the Mayer-Vietoris sequence to the following three decompositions

I. $M = (M \setminus B) \cup \overline{B}$

$$\partial (M \setminus B) \cap \overline{B} = \partial \overline{B} \cong S^{n-1},$$

II. $\overline{B} = (\overline{B} \setminus (M \setminus \partial M)) \cup \tilde{M}$

$$(\overline{B} \setminus (M \setminus \partial M)) \cap \tilde{M} = \partial \tilde{M},$$

III. $M \setminus (M \setminus \partial M) = (M \setminus B) \cup (\overline{B} \setminus (M \setminus \partial M))$

$$(M \setminus B) \cap (\overline{B} \setminus (M \setminus \partial M)) = \partial \overline{B} \cong S^{n-1}.$$
the second one to

\[ b(\bar{B} \setminus \bar{M}) + b(\bar{M}) = b(\bar{M}) + 1 \quad (2.14) \]

the third one to

\[ b(M \setminus \bar{M}) = b(M \setminus \bar{B}) + b(\bar{B} \setminus \bar{M}) - 2 \quad (2.15) \]

At last we have the equation

\[ b(\tilde{M}) = 2 b(\hat{M}) \quad (2.16) \]

because by assumption \( \tilde{M} \) can be embedded in \( B \subseteq E^n \) (cf. [9] Prop. 5.1).

Now the lemma follows directly from (2.12) - (2.16).

PROOF of Corollary B 2. Consider for example an embedding of \( S^k \times S^{n-k} \) in \( E^{n+1} \) (\( k \geq 1 \) arbitrary) as a tight hypersurface of rotation (like the standard-torus in \( E^3 \)) and change this embedding a little bit such that there is an open region \( U \) contained in some hyperplane of \( E^{n+1} \). Now define \( M \) by removing a small tight 'solid torus' of type \( S^m \times B^{n-m} \) from \( U \) \((m \geq 1)\).

By Corollary B 1 \( M \) is \((n-2)\)-tight but of course it is not tight. By suitable choice of the embedding of \( S^k \times S^{n-k} \) we started from we can assume that \( U \) lies not in the boundary of the convex hull \( \mathcal{C} \) \( M \). So we can obtain an example where \( \mathcal{O} M \) lies not in the boundary of \( \mathcal{C} \) \( M \).

REMARK. In the examples of corollary B 2 the boundary \( \mathcal{O} M \) was always tightly embedded in \( E^{n+1} \). The natural question whether there exist in higher dimensions \((n-2)\)-tight immersions with non-tight boundary seems to be open. For \( n = 2 \) an example is due to L. Rodriguez.
REFERENCES


