ON CERTAIN QUASI-COMPLEMENTED AND COMPLEMENTED BANACH ALGEBRAS

PAK-KEN WONG

Department of Mathematics
Seton Hall University
South Orange, New Jersey 07079 U.S.A.

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ABSTRACT. In this paper, we continue the study of quasi-complemented algebras and complemented algebras. The former are generalizations of the latter and were introduced in [4] and studied in [4] and [11]. Some results are proved.

KEY WORDS AND PHRASES. Quasi-complemented and complemented Banach algebras.

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1. INTRODUCTION.

Quasi-complemented algebras, which are generalizations of complemented algebras, were introduced in [4] and studied in [4] and [11]. In this paper, we continue the study of these two classes of algebras.

In Section 3, we introduce the concept of continuous quasi-complementor on a semi-simple annihilator Banach algebra. This is similar to the concept
of continuous complementor given by Alexander in [1]. Let $A$ be a simple annihilator Banach algebra such that $x \in \text{cl}_A(xA)$ for all $x$ in $A$. If $A$ is infinite dimensional, we show that every quasi-complementor on $A$ is continuous. This result is not true if $A$ is finite dimensional. In this case, we obtain that a quasi-complementor $q$ on $A$ is continuous if and only if the set $E_q$ of all $q$-projections is closed and bounded in $A$. By using these results, we give a characterization of continuous quasi-complementors (Theorem 3.4).

Section 4 is devoted to the study of uniformly continuous quasi-complementors. Let $A$ be a semi-simple annihilator Banach algebra in which $x \in \text{cl}_A(xA)$ for all $x$ in $A$ and $q$ a quasi-complementor on $A$. Suppose that $A$ has no minimal left ideals of dimension less than three. Then we show that $A$ is a dense subalgebra of some dual $B^*$-algebra $B$ and $A^q = A$ for all closed right ideals $R$ of $A$. Also every continuous complementor on $A$ is uniformly continuous.

2. NOTATION AND PRELIMINARIES.

For any subset $S$ in an algebra $A$, let $\ell_A(S)$ and $r_A(S)$ denote the left and right annihilators of $S$ in $A$, respectively. Let $A$ be a Banach algebra. Then $A$ is called an annihilator algebra, if for every closed left ideal $J$ and for every closed right ideal $R$, we have $r_A(J) = 0$ if and only if $J = A$ and $\ell_A(R) = 0$ if and only if $R = A$. If $\ell_A(r_A(J)) = J$ and $r_A(\ell_A(R)) = R$, then $A$ is called a dual algebra.

Let $A$ be a Banach algebra which is a subalgebra of a Banach algebra $B$. For each subset $S$ of $A$, $\text{cl}(S)$ (resp. $\text{cl}_A(S)$) will denote the closure of $S$ in $B$ (resp. $A$). Also $\ell(S)$ and $r(S)$ (resp. $\ell_A(S)$ and $r_A(S)$) denote the left and right annihilators of $S$ in $B$ (resp. $A$). We write $|| \cdot ||$ for the norm on $A$ and $| \cdot |$ for the norm on $B$. 
Let $A$ be a Banach algebra and let $L_r$ be the set of all closed right ideals in $A$. Following [4], we shall say that $A$ is a (right) quasi-complemented algebra if there exists a mapping $q : R \to R^q$ of $L_r$ into itself having the following properties:

1. $R \cap R^q = (0)$ (2.1)
2. $(R^q)^q = R$ (2.2)
3. If $R_1 \supseteq R_2$, then $R_{2}^q \supseteq R_{1}^q$ (2.3)

The mapping $q$ is called a (right) quasi-complementor on $A$. We know that $R + R^q$ is always dense in $A$, $A^q = (0)$ and $(0)^q = A$ (see [4]). Hence $R^q = (0)$ if and only if $R = A$.

A quasi-complemented algebra $A$ is called a (right) complemented algebra if it satisfies:

1. $R + R^q = A$ (2.4)

In this case, the mapping $q$ is called a (right) complementor on $A$ (see [6, p. 651, Definition 1]).

Let $A$ be a semi-simple Banach algebra with a quasi-complementor $q$. A minimal idempotent $f$ in $A$ is called a q-projection if $(fA)^q = (1 - f)A$. The set of all q-projection in $A$ is denoted by $E_q$. By Lemma 3.1 in [11], every non-zero right ideal of $A$ contains a q-projection.

In this paper, all algebras and linear spaces under consideration are over the complex field. Definitions not explicitly given are taken from Rickart's book [5].

We end the section with two new examples of complemented and quasi-complemented algebras.

EXAMPLE 1. Let $A$ be a dual $B^*$-algebra and $\phi$ a symmetric norming function. Then the algebra $A^{(0)}_\phi$ given in [10, p. 293] is a complemented algebra with the complementor $q : R \to (R^*)^q_{A^{(0)}_\phi}$ (Theorem 3.4 in [11]).
EXAMPLE 2. Let $G$ be an infinite compact group with the Haar measure and $A$ the algebra of all continuous functions on $G$, normed by the maximum of the absolute value and $L_1(G)$ the group algebra. It is well known that $A$ and $L_1(G)$ are dual $A^*$-algebras which are not two-sided ideals of their completions in an auxiliary norm. It is easy to see that the mapping $q : R + \xi_A(R)^*$ (resp. $R + \xi_{L_1(G)}(R)^*$) is a quasi-complementor on $A$ (resp. $L_1(G)$). However, by Theorem 3.4 in [11], $q$ is not a complementor.

3. CONTINUOUS QUASI-COMPLEMENTORS.

Let $A$ be a semi-simple annihilator Banach algebra with a quasi-complementor $q$ and $M_A$ the set of all minimal right ideals of $A$. For each $R \in M_A$, by Lemma 3.1 in [11], $R = fA$ for some $q$-projection $f$ in $A$. Therefore, $R + R^q = fA + (1 - f)A$. Let $P_R$ be the projection on $R$ along $R^q$. Then $P_R$ is continuous.

DEFINITION. Suppose $a_n \in A$ with $a_n A \in M_A$ ($n = 0, 1, 2, \ldots$). A quasi-complementor $q$ on $A$ is said to be continuous if whenever $a_n$ converges to $a_0$, then $P_{a_n A}$ converges to $P_{a_0 A}$ uniformly on any minimal left ideal of $A$.

REMARK. This is similar to the definition of continuous complementor introduced by Alexander (see [1, p. 387, Definition]).

Let $A$ be a semi-simple annihilator quasi-complemented Banach algebra such that $x \in \text{cl}_A(xA)$ for all $x$ in $A$ and $\{I_\lambda : \lambda \in \Lambda\}$ the family of all minimal closed two-sided ideals of $A$. Define $q_\lambda$ by $R^q_\lambda = R^q \cap I_\lambda$ for all closed right ideals $R$ of $I_\lambda$. Then by [4, p. 144, Theorem 3.6] $A$ is the direct topological sum of $\{I_\lambda : \lambda \in \Lambda\}$ and $q_\lambda$ is a quasi-complementor on $I_\lambda$. Let $H_\lambda$ be a minimal left ideal of $I_\lambda$. Then $H_\lambda$ is a Hilbert space under some equivalent inner product norm by [4, p. 145, Lemma 4.2]. Let $B_\lambda$ be the algebra of all completely continuous linear operators on $H_\lambda$. 

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Then by the proof of [4, p. 146, Theorem 4.3], \( I_\lambda \) is a dense subalgebra of \( B \) such that \( ||\cdot|| \) majorizes \( |\cdot| \) on \( I_\lambda \). By the proof of [8, p. 442, Lemma 5.1], \( B_\lambda \) and \( I_\lambda \) have the same socle.

**Lemma 3.1.** A quasi-complementor \( q \) on \( A \) is continuous if and only if each \( q_\lambda \) is continuous.

**Proof.** Let \( R \in M_A \) with \( R \subseteq I_\lambda_0 \) for some \( \lambda_0 \in A \). Then \( R = fA \), where \( f \) is a \( q \)-projection in \( I_\lambda_0 \). Hence, for all \( x \in A \), \( P_R(x) = fx \). If \( \lambda \neq \lambda_0 \), then \( I_\lambda I_\lambda_0 = (0) \) and so \( P_R(x) = 0 \) for all \( x \in I_\lambda \). Using this fact and the proof of [1, p. 387, Theorem 2.2], we can show that \( q \) is continuous if and only if each \( q_\lambda \) is continuous.

The following result is a generalization of [3, p. 471, Theorem 6.8].

**Lemma 3.2.** Let \( A \) be a simple annihilator Banach algebra in which \( x \in cl_A(xA) \) for all \( x \in A \). If \( A \) is infinite dimensional, then every quasi-complementor \( q \) on \( A \) is continuous.

**Proof.** Let \( H \) be a minimal left ideal of \( A \). As observed before, \( H \) is a Hilbert space under some equivalent inner product and \( A \) is a dense dual subalgebra of \( B \), the algebra of all completely continuous linear operators on \( H \).

Also \( ||\cdot|| \) majorizes \( |\cdot| \) on \( A \) and \( H \) is a minimal left ideal of \( B \). Then by [4, p. 148, Theorem 5.4], \( q \) can be extended to a quasi-complementor \( p \) on \( B \); \( M^p = cl([M \cap A]^q) \) for all closed right ideals \( M \) of \( B \). We show that \( M^p = \ell(M)^* \). In fact, let \( S(M) \) be the smallest closed subspace of \( H \) that contains the range \( x(H) \) for all \( x \in M \). Since \( ||\cdot|| \) and \( |\cdot| \) are equivalent on \( H \), it follows from [4, p. 145, Lemma 4.1] that

\[
S(M) = M \cap H = MH = (M \cap A) \cap H = (M \cap A)H. \tag{3.1}
\]

Therefore, we have

\[
S(M^p) = M^pH = cl([M \cap A]^q) \cap H = [M \cap A]^q \cap H. \tag{3.2}
\]
(see [4, p. 148] for the last equality). By the proof of [4, p. 145, Lemma 4.2],
M \cap A = c_{A}((M \cap A)HA). Since A is infinite dimensional, by [4, p. 145,
Theorem 4.2 (iii)] and (3.1)
\[
S(M) = [c_{A}(S(M)A)]^{*} \cap H = [c_{A}((M \cap A)HA))]^{*} \cap H
\]
= [M \cap A]^{*} \cap H.
Therefore, by (3.2), \( S(M) = S(M^{P}) \). Hence it follows from [3, p. 464, Lemma
4.1] and [3, p. 465, Theorem 4.2] that \( M^{P} = \ell(M)^{*} \). In particular, \( \ell \) is con-
tinuous by [1, p. 388, Theorem 2.4].

Suppose \( a_{n}A \in M_{n} \) (n = 0, 1, 2, \ldots) with \( a_{n} + o_{0} \) in \( ||\cdot|| \).
Hence \( a_{n} + o_{0} \) in \( ||\cdot|| \). Let L be a minimal left ideal of A. Then L is
a minimal left ideal of B and \( ||\cdot|| \) and \( ||\cdot|| \) are equivalent on L; also
\( a_{n}A = a_{n}B \) for all n. Let \( f_{n} \) be a (unique) q-projection contained in \( a_{n}A \).
Then \( P_{a_{n}A}(x) = f_{n}x \) for all \( x \) in A. Since \( \ell \) is continuous, \( P_{a_{0}A} \)
converges to \( P_{a_{0}A} \) uniformly on L in \( ||\cdot|| \) and hence in \( ||\cdot|| \). There-
fore \( q \) is continuous and this completes the proof.

Let A be a semi-simple annihilator quasi-complemented Banach algebra
such that \( x \in c_{A}(xA) \) for all \( x \) in A which is a dense subalgebra of a
B*-algebra B. Suppose \( ||\cdot|| \) majorizes \( ||\cdot|| \) on A. By [8, p. 442, Lemma
5.1], the set \( E \) of all hermitian minimal idempotents of B is contained
in the socle of A and so \( E \subseteq A \). Let \( E_{q} \) be the set of all q-projections
in A. For each \( e \in E \), by [4, p. 149, Lemma 6.4], there exists a unique
element \( Q(e) \in E_{q} \) such that \( Q(e)A = eA \); the mapping \( Q : e \mapsto Q(e) \) is a
one - one mapping from \( E \) onto \( E_{q} \) and is called the q-derived mapping
(see [3] and [4]).

As shown in [3, p. 475], Lemma 3.2 is not true in general, if the algebra
A is finite dimensional. In this case, we have the following result:
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LEMMA 3.3. Let $A$ be a simple finite dimensional annihilator Banach algebra with a quasi-complementor $q$ and $E_q$ the set of all $q$-projections in $A$. Then $q$ is continuous if and only if $E_q$ is a closed and bounded subset of $A$.

PROOF. By [4, p. 143, Corollary 3.2], $q$ is a complementor on $A$. Let $H$ be a minimal left ideal of $A$. Then $H$ is a Hilbert space and $A$ can be taken as the $B^*$-algebra of all linear operators on $H$. Let $Q$ be the $q$-derived mapping. By [1, p. 388, Theorem 2.4], $Q$ is continuous if and only if $q$ is continuous. Now Lemma 3.3 follows from Lemma 4.1 in [11].

We have the main result of this section.

THEOREM 3.4. Let $A$ be a semi-simple annihilator quasi-complemented Banach algebra such that $x \in cl_A (xA)$ for all $x \in A$ and let $\Lambda_0 = \{ \lambda \in \Lambda : I_\lambda \text{ is finite dimensional} \}$. Then a quasi-complementor $q$ on $A$ is continuous if and only if $E_q^\lambda$ is closed and bounded for each $\lambda \in \Lambda_0$, where $E_q^\lambda$ is the set of all $q$-projections in $I_\lambda$.

PROOF. This follows from Lemma 3.1, 3.2 and 3.3.

4. UNIFORMLY CONTINUOUS QUASI-COMPLEMENTORS.

In this section, we assume that $A$ is a semi-simple annihilator Banach algebra with a quasi-complementor $q$ such that $x \in cl_A (xA)$ for all $x \in A$. Once again, $M_A$ will be the set of all minimal right ideals of $A$ and $E_q$ the set of all $q$-projections in $A$. Also let $I_\lambda$, $H_\lambda$, $q_\lambda$, and $B_\lambda$ be as in §3. The norm on $B_\lambda$ is denoted by $|\cdot|$. 

DEFINITION. A quasi-complementor $q$ on $A$ is said to be uniformly continuous if $\{ P_{fA} : f \in E_q \}$ is closed and bounded with respect to $\| P_{fA} \|$, the operator bound norm of $P_{fA}$.

REMARK. A uniformly continuous quasi-complementor $q$ is continuous. In fact, by Theorem 3.4, we can assume that $A$ is simple and finite dimensional.
Let $H$ be a minimal left ideal of $A$. By the proof of Lemma 3.3, $A$ can be taken as the $B^*$-algebra of all linear operators on $H$. Then by [7, p. 259, Theorem 4], $E_q$ is bounded. Since $||f|| = \sup\{||fh|| : h \in H$ and $||h|| \leq 1\}$, we have $||p_{fA}|| = ||f||$ for all $f \in E_q$. It follows now that $E_q$ is closed. Hence by Theorem 3.4, $q$ is continuous.

If $u$ and $v$ are elements of a Hilbert space $H$, $u \otimes v$ will denote the operator on $H$ defined by the relation $(U \otimes v)(h) = (h, v)u$ for all $h$ in $H$.

**THEOREM 4.1.** Let $A$ be a semi-simple annihilator Banach algebra with a uniformly continuous quasi-complementor $q$ in which $x \in cl_A(xA)$ for all $x$ in $A$. Suppose that $A$ has no minimal left ideals of dimension less than three. Then $A$ is a dense subalgebra of some dual $B^*$-algebra $B$ and $R^q = \ell(R)^* \cap A$ for all closed right ideals $R$ of $A$.

**PROOF.** We know that $q$ is continuous and so is $q_\lambda$ ($\lambda \in \Lambda$). By [4, p. 148, Theorem 5.4], $q_\lambda$ induces a quasi-complementor $p_\lambda$ on $B_\lambda$. If $H_\lambda$ is finite dimensional, then by [4, p. 143, Corollary 3.2], $q_\lambda$ is a complementor and so by the proof of Theorem 4.3 in [11], $p_\lambda$ has the form $J_\lambda^{p_\lambda} = \ell(J_\lambda)^*$ for all closed right ideals $J_\lambda$ in $B_\lambda$. If $H_\lambda$ is infinite dimensional, this is also true by the proof of Lemma 3.2.

We show that there exists a constant $M$ such that

$$||h|| \leq |h| \leq M|h||$$

$h \in H_\lambda$, $\lambda \in \Lambda$. (4.1)

We follow the argument in [1, p. 393, Lemma 4.3]. It can be assumed that

$$||h|| \leq |h| \leq \sqrt{2}|h||$$

$h \in H_\lambda$, $\lambda \in \Lambda$. (4.2)

Suppose (4.1) does not hold. Then there exists $x_n$ in $H_n$ such that

$$||x_n|| = 1$$

and $|x_n| = k > n$. By (4.2), we can find $z_n$ in $H_n$ such that

$$||z_n|| = 1, ||z|| \leq \sqrt{2}$.

Write $z_n = a_n x_n + x'_n$ with $a_n \in C$, $x'_n \in H_n$ and $(x_n, x'_n) = 0$. Put $y_n = k^{-1}x_n + x'_n$ and $f_n = (y_n \otimes y_n)/(y_n, y_n)$. Then $f_n \in E_q$ and
\[ \left\| \mathbf{f}_n \mathbf{A} \left( \mathbf{x}_n \right) \right\| = \left\| \mathbf{y}_n \otimes \frac{\mathbf{y}_n \mathbf{x}_n}{\left( \mathbf{y}_n, \mathbf{y}_n \right)} \right\| = \left\| \frac{\left( \mathbf{x}_n, \mathbf{y}_n \right)}{\left( \mathbf{y}_n, \mathbf{y}_n \right)} \right\| \mathbf{y}_n \rightarrow \infty. \]

Hence \( \left\{ \left\| \mathbf{f}_n \mathbf{A} \right\| \right\} \) is unbounded and this contradicts the uniform continuity of \( \mathbf{q} \). Therefore (4.1) holds. Now by using the argument in Theorem 4.3, in [11], we can complete the proof.

Theorem 4.1 shows that there is essentially one type of uniformly continuous quasi-complementors on \( \mathbf{A} \).

The following result generalizes [4, p. 153, Theorem 7.6].

**COROLLARY 4.2.** Let \( \mathbf{A} \) and \( \mathbf{B} \) be as in Theorem 4.1. Then \( \mathbf{q} \) is a complementor on \( \mathbf{A} \) if and only if \( \mathbf{A} \) is a left ideal of \( \mathbf{B} \).

**PROOF.** This follows from Theorem 4.1 and Theorem 3.4 in [11].

On the other hand, if \( \mathbf{q} \) is a complementor, then we have:

**THEOREM 4.3.** Let \( \mathbf{A} \) be a semi-simple annihilator Banach algebra such that \( \mathbf{A} \) has no minimal left ideal of dimension less than three. Then every continuous complementor \( \mathbf{q} \) on \( \mathbf{A} \) is uniformly continuous.

**PROOF.** By [6, p. 655, Theorem 4], \( \mathbf{A} \) is the direct topological sum of its minimal closed two-sided ideals \( \{ \mathbf{I}_\lambda : \lambda \in \Lambda \} \) each of which is a complemented and dual algebra. Let \( \mathbf{q}_\lambda, \mathbf{H}_\lambda \) and \( \mathbf{B}_\lambda \) be as before and \( \| \cdot \| \) the norm on \( \mathbf{B}_\lambda \).

By [1, p. 390, Theorem 3.2], \( \mathbf{q}_\lambda \) induces a complementor \( \mathbf{p}_\lambda \) on \( \mathbf{B}_\lambda \) and by [1, p. 391, Theorem 3.3], \( \mathbf{p}_\lambda \) has the form \( \mathbf{J}_\lambda = \mathbf{k}(\mathbf{J}_\lambda)^* \) for all closed right ideals \( \mathbf{J}_\lambda \) in \( \mathbf{B}_\lambda \). By [1, p. 393, Lemma 4.3], there exists a constant \( M \) such that

\[ \| h \| \leq \| h \| \leq M \| h \| \quad (h \in \mathbf{H}_\lambda, \lambda \in \Lambda). \tag{4.3} \]

Let \( \mathbf{B} \) be the \( \mathbf{B}^*(\infty) \)-sum of \( \{ \mathbf{B}_\lambda : \lambda \in \Lambda \} \). Then \( \mathbf{B} \) is a dual \( \mathbf{B}^* \)-algebra and \( \mathbf{E} \) coincides with the set of all hermitian minimal idempotents in \( \mathbf{B} \). Since \( \mathbf{q} \) is a left ideal of \( \mathbf{B} \), it is well-known that there exists a constant \( k \) such that \( \| \mathbf{b} \| \leq k \| \mathbf{b} \| \| \mathbf{a} \| \) for all \( \mathbf{b} \) in \( \mathbf{B} \) and \( \mathbf{a} \) in \( \mathbf{A} \). Then
\[ | | P f_A(x) | | = | | f(x) | | \leq k | | f | | \quad | | x | | = k | | x | | \quad \text{for all } x \text{ in } A \text{ and } f \text{ in } E \]

Hence \( \{ P f_A : f \in E \} \) is bounded. It remains to show that it is closed. Let \( \{ P f_n A \} \) be a Cauchy sequence, where \( f_n \in E \). We show that, for \( m \) and \( n \) large enough, \( f_m \) and \( f_n \) are contained in the same minimal closed two-sided ideal. Suppose this is not so. Then there exists some minimal closed two-sided ideal \( I_{\lambda_n} \) of \( A \) such that \( f_n \in I_{\lambda_n} \), but \( f_m \notin I_{\lambda_n} \). Let \( H_{\lambda_n} \) be the minimal left ideal in \( I_{\lambda_n} \). Since \( | f_n | = 1 \), we can choose \( h_n \in H_{\lambda_n} \) such that \( | f_n h_n | > 1/2 \) with \( | h_n | = 1 \). Since \( f_m I_{\lambda_n} = (0) \), by (4.3) we have

\[
\frac{1}{2} < | f_n h_n | = | f_n h_n - f_m h_n | \leq M | f_n h_n - f_m h_n | \leq M | P f_n A - P f_m A | | h_n | = M | P f_n A - P f_m A | .
\]

But \( \{ P f_n A \} \) is a Cauchy sequence; a contradiction. Therefore, we can assume that \( f_m \) and \( f_n \) belong to the same \( I_{\lambda_n} \). Hence,

\[
| f_n - f_m | = \sup \{ | (f_n - f_m) h | : h \in H_{\lambda_n} \text{ and } | h | \leq 1 \} \leq M | P f_n A - P f_m A | .
\]

and so \( \{ f_n \} \) is a Cauchy sequence in \( | | . | | \). Since \( E \) is closed in \( | | . | | \) by Theorem 4.2, in [11], \( f_n \to f \) in \( | | . | | \) for some \( f \in E \). Since

\[
| | (P f_n A - P f_A)(x) | | = | | f_n x - f x | | \leq k | | f_n - f | | | | x | | \quad \text{for all } x \text{ in } A,
\]

\( P f_n A \to P f_A \) and so \( \{ P f_A : f \in E \} \) is closed. This completes the proof.
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