A NOTE ON STRICTLY CYCLIC SHIFTS ON ℓ_p

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ABSTRACT. In this paper the author shows that a well known sufficient condition for strict cyclicity of a weighted shift on ℓ_p is not a necessary condition for any p with $1 < p < \infty$.

1. INTRODUCTION.

For $1 \leq p < \infty$ let ℓ_p be the Banach space of absolutely p -summable sequences of complex numbers. Let S_α denote the weighted shift on ℓ_p with weight sequence $\alpha = \{\alpha_n\}_1^\infty$ defined by $S_\alpha \left[\sum_{n=0}^\infty x_n e_n \right] = \sum_{n=1}^\infty \alpha_n x_{n-1} e_n$. Let $\beta_0 = 1$ and $\beta_n = \alpha_1 \alpha_2 \dots \alpha_n$ for all $n \geq 1$. (For more detail we refer the reader to [2] and [3]).

Mary Embry [1] showed that for $p = 1$ the weighted shift S_α is strictly cyclic if and only if

$$\sup_{n,m} \left| \frac{\beta_{n+m}}{\beta_n \beta_m} \right| < \infty. \quad (1.1)$$

Edward Kerlin and Alan Lambert [2] considered the natural extension to (1.1) for the case $1 < p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\sup_n \sum_{m=0}^n \left| \frac{\beta_n}{\beta_m \beta_{n-m}} \right|^q < \infty \quad (1.2)$$

and showed (1.2) implies S_α is strictly cyclic on ℓ_p . They also proved that (1.2) is necessary if the weight sequence α is eventually decreasing.

This strongly suggested that (1.2) is a necessary and sufficient condition for strict cyclicity. However, we will show in this paper that (1.2) is not a necessary condition for S_α to be strictly cyclic for any p with $1 < p < \infty$.

2. PROOF.

To preserve the clarity of the proof we will consider the cases $1 < p \leq 2$ and $2 < p < \infty$ separately.

(a) Let $1 < p \leq 2$ and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $\{n_k\}_1^\infty$ be a sequence of rapidly increasing positive integers, e.g., choose $n_1 = 10$ and $n_k = (10n_{k-1})^{10qn_{k-1}}$ for $k > 1$.

We now define the weight sequence $\alpha = \{\alpha_i\}_1^\infty$ by $\alpha_1 = \alpha_2 = \dots = \alpha_{n_1} = 1$ and for $k > 1$

$$\alpha_i = \begin{cases} n_k^{-1} & \text{if } n_{k-1} < i \leq n_k - n_{k-1} \\ n_{k-1}^{-1} & \text{if } n_k - n_{k-1} < i \leq n_k. \end{cases}$$

Clearly, $\alpha_{n_{k-1}+1} = \alpha_{n_{k-1}+2} = \dots = \alpha_{n_k - n_{k-1}}$ and $\alpha_{n_k - \ell + 1} \leq \alpha_\ell$ for $1 \leq \ell \leq \frac{n_k}{2}$.

Thus, for $0 < m < n_k$

$$\frac{\beta_{n_k}}{\beta_m \beta_{n_k-m}} = \frac{\alpha_{n_k-m+1} \alpha_{n_k-m+2} \dots \alpha_{n_k}}{\alpha_1 \alpha_2 \dots \alpha_m} \geq \frac{\alpha_{n_k - n_{k-1} + 1} \dots \alpha_{n_k}}{\alpha_1 \alpha_2 \dots \alpha_{n_{k-1}}} \geq (\alpha_{n_k})^{n_{k-1}}.$$

Therefore,

$$\sum_{m=1}^{n_k-1} \left| \frac{\beta_{n_k}}{\beta_m \beta_{n_k-m}} \right|^q \geq n_k (\alpha_{n_k})^{qn_{k-1}} = n_k n_{k-1}^{-qn_{k-1}} \rightarrow \infty \text{ as } k \rightarrow \infty.$$

Hence,

$$\sup_n \sum_{m=0}^n \left| \frac{\beta_n}{\beta_m \beta_{n-m}} \right|^q = \infty.$$

We now show that S_α is strictly cyclic. It is known that $S_\alpha[2]$ is strictly cyclic if and only if

$$\sum_{n=0}^{\infty} \left| \sum_{m=0}^n \frac{\beta_n}{\beta_m \beta_{n-m}} x_m y_{n-m} \right|^p < \infty \text{ for all } x, y \in \ell_p. \quad (1.3)$$

Obviously, for $0 < m < n$

$$\frac{\beta_n}{\beta_m \beta_{n-m}} = \frac{\alpha_{n-m+1} \cdots \alpha_n}{\alpha_1 \alpha_2 \cdots \alpha_m} \leq \alpha_n$$

and $\frac{\beta_n}{\beta_m \beta_{n-m}} = 1$ for $m = 0$ or $m = n$.

Thus,

$$\begin{aligned} \sum_{n=0}^{\infty} \left| \sum_{m=0}^n \frac{\beta_n}{\beta_m \beta_{n-m}} x_m y_{n-m} \right|^p &\leq \sum_{n=0}^{\infty} \left\{ |x_0 y_n| + |y_0 x_n| + \sum_{m=1}^{n-1} \alpha_n |x_m y_{n-m}| \right\}^p \\ &\leq \sum_{n=0}^{\infty} \left\{ |x_0 y_n| + |y_0 x_n| + \alpha_n \|x\|_2 \|y\|_2 \right\}^p < \infty \\ &\text{since } \alpha, x, y \in \ell_p. \end{aligned}$$

Hence (1.3) holds and S_α is strictly cyclic.

(b) Let $2 < p < \infty$ and let $\frac{1}{p} + \frac{1}{q} = 1$. Let $\{n_k\}_1^\infty$ be a sequence of rapidly increasing integers, e.g., choose $n_1 = 10$ and n_k such that

$$\sum_{n=1}^{n_k} \frac{1}{n} \geq (10n_{k-1})^{10n_{k-1}} \text{ for } k > 1.$$

Define $\{d_i\}_1^\infty$ such that $\prod_{i=1}^n d_i = n^{-\frac{1}{q}}$ for $n = 1, 2, \dots$ and define $\{s_k\}_1^\infty$ by $s_1 = 10$ and $s_k = 2s_{k-1} + 2n_k$ for $k > 1$.

We now define the weight sequence $\alpha = \{\alpha_i\}_1^\infty$ by $\alpha_1 = \dots = \alpha_{s_1} = 1$ and for $k > 1$,

$$\alpha_i = \begin{cases} n_k^{-2} & \text{if } s_{k-1} < i \leq s_{k-1} + n_k \\ n_k^{-2} d_{s_k - s_{k-1} - i + 1} & \text{if } s_{k-1} + n_k < i \leq s_k - s_{k-1} \\ n_{k-1}^{-2} & \text{if } s_k - s_{k-1} < i \leq s_k. \end{cases}$$

Now, for $s_{k-1} < m < \frac{s_k}{2}$,

$$\begin{aligned} \frac{\beta_{s_k}}{\beta_m \beta_{s_k - m}} &= \frac{\alpha_{s_k - m + 1} \dots \alpha_{s_k}}{\alpha_1 \alpha_2 \dots \alpha_m} \geq \frac{\alpha_{s_k - m + 1} \dots \alpha_{s_k - s_{k-1}}}{\alpha_{s_{k-1} + 1} \dots \alpha_m} n_{k-1}^{-2s_{k-1}} \geq n_{k-1}^{-2s_{k-1}} \prod_{i=1}^{m-s_{k-1}} d_i \\ &= n_{k-1}^{-2s_{k-1}} (m - s_{k-1})^{-\frac{1}{q}}. \end{aligned}$$

Thus,

$$\sum_{m=0}^{s_k} \left| \frac{\beta_{s_k}}{\beta_m \beta_{s_k - m}} \right|^q \geq n_{k-1}^{-2qs_{k-1}} \sum_{i=1}^{n_k} i^{-1} \geq n_{k-1}^{-10n_{k-1}} \sum_{i=1}^{n_k} i^{-1} \rightarrow \infty \text{ as } k \rightarrow \infty.$$

Hence,

$$\sup_n \sum_{m=0}^n \left| \frac{\beta_n}{\beta_m \beta_{n-m}} \right|^q = \infty.$$

We now show that

$$\sum_{n=0}^{\infty} \left| \sum_{m=0}^n \frac{\beta_n}{\beta_m \beta_{n-m}} x_m y_{n-m} \right|^p < \infty \text{ for all } x, y \in \ell_p.$$

If $s_{k-1} < n < s_k - s_{k-1}$ and $0 < m < n$ then,

$$\frac{\beta_n}{\beta_m \beta_{n-m}} \leq n_k^{-2}.$$

Let $h = \min\{m, n-m\}$ then, for $s_k - s_{k-1} \leq n \leq s_k$ and $0 < m < n$,

$$\frac{\beta_n}{\beta_m \beta_{n-m}} \leq n_{k-1}^{-2} h^{-\frac{1}{q}}.$$

Thus,

$$\sum_{k=2}^{\infty} \sum_{n=s_{k-1}+1}^{s_k-s_{k-1}-1} \left| \sum_{m=1}^{n-1} \frac{\beta_n}{\beta_m \beta_{n-m}} x_m y_{n-m} \right|^p \leq \sum_{k=2}^{\infty} \sum_{n=s_{k-1}+1}^{s_k-s_{k-1}-1} \left(n_k^{-2} \sum_{m=1}^{n-1} |x_m y_{n-m}| \right)^p < \infty. \quad (1.4)$$

Let $\delta = \frac{p}{p-2}$ then $\delta > q$ and $\frac{1}{\delta} + \frac{2}{p} = 1$. Let $M = \left(\sum_{m=1}^{\infty} 2m^{-\frac{\delta}{q}} \right)^{\frac{1}{\delta}} \left(\|x\|_p^p + \|y\|_p^p \right)^{\frac{2}{p}} < \infty$.

Then,

$$\begin{aligned} & \sum_{k=2}^{\infty} \sum_{n=s_k-s_{k-1}}^{s_k} \left| \sum_{m=1}^{n-1} \frac{\beta_n}{\beta_m \beta_{n-m}} x_m y_{n-m} \right|^p \\ & \leq \sum_{k=2}^{\infty} \sum_{n=s_k-s_{k-1}}^{s_k} \left(\sum_{m=1}^{\infty} n_{k-1}^{-2} h^{-\frac{1}{q}} |x_m y_{n-m}| \right)^p, \text{ where } h = \min\{m, n-m\} \\ & \leq \sum_{k=2}^{\infty} \sum_{n=s_k-s_{k-1}}^{s_k} n_{k-1}^{-2p} M^p < \infty. \end{aligned}$$

Combining (1.4) and (1.5) we obtain that (1.3) is satisfied. QED

References

1. Embry, Mary. Strictly Cyclic Operator Algebras on a Banach Space, to appear in Pac. J. Math.
2. Kerlin, Edward and Alan Lambert. Strictly Cyclic Shifts on ℓ_p , Acta Sci. Math. 35 (1973) 87-94.
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