GRAPHS WHICH HAVE PANCYCLIC COMPLEMENTS

H. JOSEPH STRAIGHT

Department of Mathematics
SUNY College at Fredonia
Fredonia, New York 14063

(Received January 23, 1978 and in revised form March 1978)

ABSTRACT. Let p and q denote the number of vertices and edges of a graph G , respectively. Let $\Delta(G)$ denote the maximum degree of G , and $\bar{G}$ the complement of G . A graph G of order p is said to be pancyclic if G contains a cycle of each length n , $3 \leq n \leq p$. For a nonnegative integer k , a connected graph G is said to be of rank k if $q = p - 1 + k$. (For k equal to 0 and 1 these graphs are called trees and unicyclic graphs, respectively.)

In 1975, I posed the following problem: Given k , find the smallest positive integer p_k , if it exists, such that whenever G is a rank k graph of order $p \leq p_k$ and $\Delta(G) < p - 2$ then $\bar{G}$ is pancyclic. In this paper it is shown that a result by Schmeichel and Hakimi (2) guarantees that p_k exists. It is further shown that for $k = 0, 1$, and 2 , $p_k = 5, 6$, and 7 , respectively.

1. INTRODUCTION.

Throughout this paper the terminology of Behzad and Chartrand (1) will be

followed. In particular, p and q shall denote the number of vertices and edges of a graph G , respectively. We let $\Delta(G)$ denote the maximum degree of G and $\overline{G}$ denote the complement of G .

A graph G of order p is called pancyclic if G contains a cycle of each length n , $3 \leq n \leq p$. For a nonnegative integer k , a connected graph G is said to be of rank k if $q = p - 1 + k$. Here the number k gives the number of independent cycles in G . When k equals 0 or 1 these graphs are called trees or unicyclic graphs, respectively.

In this paper we explore the following idea: if G is a graph having, in some sense, little cycle structure relative to its order, then perhaps $\overline{G}$ will have a great deal of cycle structure. As an example, consider the graph shown in Figure 1. This graph is a tree, i.e., a connected graph having no cycles. On the other hand note that its complement is pancyclic.

Figure 1. A tree...and its pancyclic complement

In 1975, after obtaining the results for $k = 0, 1$, and 2 which are presented here, I posed the following problem: Given k , find the smallest positive integer p_k , if it exists, such that whenever G is a graph of rank k of order $p \geq p_k$ and $\Delta(G) < p - 2$, then $\overline{G}$ is pancyclic. Recently, J. A. Bondy has pointed out that the existence of p_k is guaranteed by the following result due to Schmeichel and Hakimi [2].

THEOREM. Let G be a graph with p vertices, q edges, and minimum degree $\delta \geq 2$. If

$$q > \begin{cases} \frac{1}{2}(p^2 - (2\delta + 1)p + 3\delta^2 + \delta), & 2 \leq \delta \leq \frac{p+5}{6} \text{ and } p \text{ odd} \\ \frac{1}{2}(p^2 - (2\delta + 1)p + 3\delta^2 + \delta), & 2 \leq \delta \leq \frac{p+8}{6} \text{ and } p \text{ even} \\ \frac{1}{8}(3p^2 - 8p + 5) + \delta, & \frac{p+5}{6} \leq \delta \leq \frac{p-1}{2} \text{ and } p \text{ odd} \\ \frac{1}{8}(3p^2 - 10p + 16) + \delta, & \frac{p+8}{6} \leq \delta \leq \frac{p-2}{2} \text{ and } p \text{ even} \\ \frac{1}{4}p^2, & \frac{p-1}{2} < \delta \end{cases}$$

then G is pancyclic.

COROLLARY. Let k be a nonnegative integer. Then there exists a positive integer p_k such that whenever G is a graph of rank k of order $p \geq p_k$ and $\Delta(G) < p - 2$, then $\bar{G}$ is pancyclic.

PROOF. Let G be a graph of rank k with $\Delta(G) < p - 2$. If G has p vertices, then $\bar{G}$ has p vertices, $q = \frac{p^2 - 3p + 2}{2} - k$ edges and minimum degree $\delta \geq 2$. Depending on the value of δ , the requirements for q given by the theorem would yield the following inequalities:

$$(2\delta - 2)p > 3\delta^2 + \delta + 2k - 2$$

$$p^2 - 4p > 8k + 8\delta - 3$$

$$p^2 - 2p > 8k + 8\delta + 8$$

$$p^2 - 6p > 4k - 4$$

Note that each of the above inequalities is true provided that p is large enough. Hence we can choose p_k to be the least positive integer which makes all the above inequalities true.

The above theorem yields an upper bound for p_k ; however, in the known cases it does not give us a very good bound. For example, the theorem

would tell us that $p_2 \leq 10$, whereas we will show that $p_2 = 7$. In the remainder of the paper we show that for $k = 0, 1$, and 2 , $p_k = 5, 6$, and 7 , respectively.

2. MAIN RESULTS.

THEOREM 1. If G is a tree of order $p \geq 5$ with $\Delta(G) < p - 2$, then $\overline{G}$ is pancyclic.

PROOF. The proof is by induction on p . If $p = 5$, then G is a path on 5 vertices. Thus $\overline{G} = C_5 + e$ and so clearly $\overline{G}$ is pancyclic. Now let $p \geq 6$, and assume the result holds for all trees of order less than p . Let G be a tree of order p with $\Delta(G) < p - 2$. If $\Delta(G) < p - 3$, let v be an end vertex of G . If $\Delta(G) = p - 3$, then unless G is the graph of Figure 1, we may choose v to be an end vertex adjacent with the unique vertex of degree $p - 3$. Now consider $G - v$, which is a tree of order $p - 1$ with $\Delta(G - v) < (p - 1) - 2$. Hence by the induction hypothesis, $\overline{G - v}$ has a cycle of each length n , $3 \leq n \leq p - 1$. Therefore so does $\overline{G}$. Since $\deg_G v = p - 2 > \frac{p-1}{2}$, v must be adjacent in $\overline{G}$ to two consecutive vertices on the $(p-1)$ -cycle in $\overline{G - v}$. Thus this cycle can be extended in $\overline{G}$ to a cycle of length p . Therefore, $\overline{G}$ is pancyclic. Now by induction the proof is completed.

COROLLARY 1. If G is a forest of order $p \geq 5$ with $\Delta(G) < p - 2$, then $\overline{G}$ is pancyclic.

PROOF. Note that there exists a tree H with $\Delta(H) < p - 2$ containing G as a spanning subgraph. Since $\overline{H}$ is pancyclic and $\overline{H} \subseteq \overline{G}$, $\overline{G}$ is pancyclic.

THEOREM 2. If G is a unicyclic graph of order $p \geq 6$ with $\Delta(G) < p - 2$, then $\overline{G}$ is pancyclic.

PROOF. Let $u_1, u_2, \dots, u_n$ denote the cycle vertices of G . Among the vertices u_i , we will choose one of minimum degree in G ; call it u .

CASE 1. Suppose $n \geq 4$. Then $\deg u \leq 2 + \frac{p-4}{4} = \frac{p+4}{4}$. Note that $\frac{p+4}{4} < \frac{p-1}{2}$ provided that $p \geq 6$. If $\Delta(G) = p-3$, notice that we can choose u so that $\Delta(G-u) = p-4$ unless G is the graph of Figure 3a. Now since $G-u$ is a forest, $\overline{G-u}$ is pancyclic by Corollary 1. Since u is adjacent in $\overline{G}$ to two consecutive vertices on the $(p-1)$ -cycle of $\overline{G-u}$, this cycle can be extended to a p -cycle in $\overline{G}$. Therefore, $\overline{G}$ is pancyclic.

CASE 2. Suppose $n = 3$. First, suppose that if $\Delta(G) = p-3 = \deg v$, then v is one of the 3 cycle vertices. Then we know that $\Delta(G-u) \leq p-4$. Now $\deg u \leq 2 + \frac{p-3}{3} = \frac{p+3}{3}$. If $\deg u < \frac{p-1}{2}$ we can proceed just as in Case 1. This will happen if $p > 9$ or if $p = 8$. If it does not happen then G must be one of the graphs shown in Figures 3b - d, all of which have pancyclic complements. Secondly suppose that $\Delta(G) = p-3 = \deg_G v$ and v is not a cycle vertex. Then G is the graph of Figure 2.

Figure 2

If $p \geq 8$, we can remove u and proceed as above. If $p = 6$ or 7 , we again get special case graphs, which can be shown to have pancyclic complements.

Figure 3

COROLLARY 2. If G is a graph of order $p \geq 6$ with $\Delta(G) < p - 2$ and G contains exactly one cycle, then $\bar{G}$ is pancyclic.

The five-cycle C_5 is a unicyclic graph on 5 vertices which does not have a pancyclic complement. This shows that $p_1 = 6$. The graph shown in Figure 4 is a rank 2 graph of order 6 whose complement does not contain a 3-cycle. Hence $p_2 \geq 7$. Our final result shows that indeed $p_2 = 7$.

Figure 4

THEOREM 3. If G is a graph of rank 2 of order $p \geq 7$ with $\Delta(G) < p - 2$, then $\overline{G}$ is pancyclic.

PROOF. We consider three cases.

CASE 1. G has a cycle with a diagonal. Let $u_1, u_2, \dots, u_n$ be the cycle vertices of G , $n \geq 4$, and suppose u_1u_i is the diagonal of the cycle, $3 \leq i \leq n - 1$. First, suppose $\Delta(G) < p - 3$. Choose a cycle vertex u which has the smallest degree in G among the cycle vertices. Then $\deg u \leq \frac{p+6}{4}$. Note that $\frac{p+6}{4} < \left\{ \frac{p-1}{2} \right\}$ if $p \geq 8$. Also there does not exist a rank 2 graph on 7 vertices with $\Delta(G) = 3$ and $\deg u_i = 3$, $1 \leq i \leq n$. Thus $p \geq 7$ implies $\deg_G u < \left\{ \frac{p-1}{2} \right\}$. Now by Corollary 2, $\overline{G - u}$ is pancyclic. Since u is adjacent in $\overline{G}$ to more than half the other vertices, $\overline{G}$ is pancyclic. Secondly, suppose $\Delta(G) = p - 3$ and $p \geq 8$. Then we either choose u as above or of degree 3 in G in such a way that $\Delta(G - u) = p - 4$. Since $3 < \frac{p-1}{2}$, $\deg_G u < \frac{p-1}{2}$ and so we argue as before. Lastly we must consider the case where $p = 7$, $\Delta(G) = 4$, and there does not exist a vertex u with $\deg_G u = 2$ and $\Delta(G - u) = 3$. In this case G must be one of the graphs shown in Figure 5, all of which have pancyclic complements.

Figure 5

CASE 2. G has the following configuration as a subgraph.

Again let u be a cycle vertex of smallest degree. Then

$$\deg_G u \leq \begin{cases} 2 + \frac{p-3}{5}, & u \neq v \\ 4 + \frac{p-13}{5}, & u = v \end{cases} = \frac{p+7}{5} < \frac{p-1}{2}.$$

Also if $\Delta(G) = p - 3$, then clearly it is possible to choose u so that

$\Delta(G - u) = p - 4$. Hence we may argue as before.

CASE 3. G contains the configuration of Figure 6.

Figure 6

Choose u as before. Then

$$\deg_G u \leq \begin{cases} 2 + \frac{p-4}{6}, & u \notin \{v_1, v_2\} \\ 3 + \frac{p-10}{6}, & u \in \{v_1, v_2\} \end{cases} = \frac{p+8}{6} < \frac{p-1}{2}.$$

If $\Delta(G) = p - 3$, then $\deg v_i = p - 3$ for $i = 1$ or 2 but not both, and all other cycle vertices have degree 2. Hence we may choose u so that $\Delta(G - u) = p - 4$ and proceed as before.

REFERENCES

1. Behzad, M. and G. Chartrand. Introduction to the Theory of Graphs, Allyn and Bacon, Boston, 1971.
2. Schmeichel, E. F. and S. L. Hakimi. Pancyclic Graphs and a Conjecture of Bondy and Chvatal, J. Combinatorial Theory (B) 17 (1974) 22-34.

KEY WORDS AND PHRASES. *Graphs, pancyclic graphs, and unicyclic graphs.*

AMS(MOS) SUBJECT CLASSIFICATION (1970) CODES. 05C99.