EQUIVALENCE CLASSES OF THE 3RD GRASSMAN SPACE OVER A 5-DIMENSIONAL VECTOR SPACE

KULDIP SINGH

Department of Mathematics
The University of New Brunswick
Fredericton, N.B., Canada E3B 5A3

(Received June 10, 1977)

ABSTRACT. An equivalence relation is defined on $\Lambda^r V$, the r^{th} Grassman space over V and the problem of the determination of the equivalence classes defined by this relation is considered. For any r and V , the decomposable elements form an equivalence class. For $r = 2$, the length of the element determines the equivalence class that it is in. Elements of the same length are equivalent, those of unequal lengths are inequivalent. When $r \geq 3$, the length is no longer a sufficient indicator, except when the length is one. Besides these general questions, the equivalence classes of $\Lambda^3 V$, when $\dim V = 5$ are determined.

KEY WORDS AND PHRASES. Grassman space, equivalent classes, representation of equivalent classes.

AMS (MOS) SUBJECT CLASSIFICATION (1970) CODES. 14M15.

Suppose V is a finite dimensional vector space over an arbitrary field F and r is a positive integer. Consider $\Lambda^r V$, the r th Grassman space over V . We define an equivalence relation on $\Lambda^r V$ as follows: If X and Y are in $\Lambda^r V$, we write $X \sim Y$ iff $\exists$ a non-singular linear transformation $T: V \rightarrow V$ such that $C_r(T)X = Y$, where $C_r(T)$ is the r th exterior product of T . Using the facts, that if T and S are two linear transformations of V , then $C_r(T)C_r(S) = C_r(TS)$ and if T is non-singular, then $C_r(T^{-1}) = C_r(T)^{-1}$, it follows that the above relation is an equivalence relation.

We consider the problem of determining the number of equivalence classes, into which the set $\Lambda^r V$ is decomposed, along with a system of distinct representatives of these equivalence classes.

DEFINITIONS. 1. If $X \in \Lambda^r V$ and $X = x_1 \wedge \dots \wedge x_r$, we say X is decomposable.

2. If $X \in \Lambda^r V$, we define its length, to be denoted by $\ell(X)$ as $\ell(X) = \min\{m \mid X \text{ is a sum of } m \text{ decomposable elements of } \Lambda^r V\}$.

3. If $X \in \Lambda^r V$, we define a subspace $[X]$ of V as $[X] = \cap\{U \mid U \text{ is a subspace of } V \text{ and } X \in \Lambda^r U\}$.

4. If $X \in \Lambda^r V$, we define the rank of X to be denoted by $\rho(X)$ as $\rho(X) = \dim[X]$.

PROPOSITION 1. If $X, Y \in \Lambda^r V$ and $X \sim Y$, then (i) $\ell(X) = \ell(Y)$,
(ii) $P(X) = P(Y)$.

PROOF. (i) Let $T: V \rightarrow V$ be a n.s.l.t. such that $C_r(T)X = Y$.
If $\ell(X) = s$ $X = \sum_{i=1}^s X_i$, where $X_i \in \Lambda^r V$ and $\ell(X_i) = 1$.

Then $Y = C_r(T)X = \sum_{i=1}^s C_r(T)X_i$. This implies $\ell(Y) \leq s = \ell(X)$. Similarly

$Y \sim X$ implies $\ell(Y) \leq \ell(X)$ and this proves (i).

(ii) We first remark that if U and W are subspaces of V , then $X \in \Lambda^r U$ implies $Y \in \Lambda^{rT}(U)$ and $Y \in \Lambda^r W$ implies $X \in \Lambda^{rT^{-1}}(W)$, where $T: V \rightarrow V$ is a n.s.l.t. such that $Y = C_r(T)X$. From this remark, it follows easily that $[Y] = T[X]$ and hence $P(X) = P(Y)$.

PROPOSITION 2. If U and W are subspaces of V , then $\Lambda^r U \cap \Lambda^r W = \Lambda^r(U \cap W)$.

PROOF. Clearly $\Lambda^r(U \cap W) \subseteq (\Lambda^r U) \cap (\Lambda^r W)$. To prove the inclusion in the other direction, let $x_1, x_2, \dots, x_k$ be a basis of $U \cap W$ and extend it to a basis $x_1, \dots, x_k, y_1, \dots, y_s$ of U and a basis $x_1, \dots, x_k, z_1, \dots, z_t$ of W . Then $x_1, \dots, x_k, y_1, \dots, y_s, z_1, \dots, z_t$ is a basis of $U + W$. If $A = \{x_i \wedge x_j \mid 1 \leq i < j \leq k\}$, $B = \{y_i \wedge y_j \mid 1 \leq i < j \leq s\}$, $C = \{z_i \wedge z_j \mid 1 \leq i < j \leq t\}$, $D = \{x_i \wedge y_j \mid 1 \leq i \leq k; 1 \leq j \leq s\}$, $E = \{x_i \wedge z_j \mid 1 \leq i \leq k; 1 \leq j \leq t\}$, $F = \{y_i \wedge z_j \mid 1 \leq i \leq s; 1 \leq j \leq t\}$, then the sets $A, A \cup B \cup D, A \cup C \cup E$, and $A \cup B \cup C \cup D \cup E \cup F$ form bases of $\Lambda^r(U \cap W), \Lambda^r U, \Lambda^r W$ and $\Lambda^r(U + W)$ respectively. If $X \in (\Lambda^r U) \cap (\Lambda^r W)$, then

$$X = \sum_A a_{ij} x_i \wedge x_j + \sum_B b_{ij} y_i \wedge y_j + \sum_D d_{ij} x_i \wedge y_j \quad \text{and also}$$

$$X = \sum_A a_{ij} x_i \wedge x_j + \sum_C c_{ij} z_i \wedge z_j + \sum_E e_{ij} x_i \wedge z_j. \quad \text{Hence } a_{ij} = a_{ij} \text{ and}$$

$b_{ij} = d_{ij} = c_{ij} = e_{ij} = 0$ for all the appropriate values of the indices i and j . Thus $X \in \Lambda^r(U \cap W)$.

REMARK 1. The result of Proposition 2 holds for any number of subspaces of V .

REMARK 2. If $X \in \Lambda^r V$ and $\mathcal{S} = \{U \mid U \text{ is a subspace of } V, X \in \Lambda^r U\}$, then $\Lambda^r[X] = \Lambda^r(\bigcap_{U \in \mathcal{S}} U) = \bigcap_{U \in \mathcal{S}} (\Lambda^r U)$. Thus $X \in \Lambda^r[X]$ and $[X]$ is the smallest such subspace of V .

PROPOSITION 3. Let $X \in \Lambda^2 V$, $\ell(X) = k$ and $X = \sum_{i=1}^k x_i \wedge y_i$, then $x_1, \dots, x_k, y_1, \dots, y_k$ are linearly independent.

PROOF. If not, then one of them (say) y_k is a linear combination of the

remaining $x_1, \dots, x_k, y_1, \dots, y_{k-1}$. Let $y_k = \sum_{i=1}^k a_i x_i + \sum_{j=1}^{k-1} b_j y_j$.

Then $x_k \wedge y_k = \sum_{i=1}^k a_i x_k \wedge x_i + \sum_{j=1}^{k-1} b_j x_k \wedge y_j$. Hence X can be written as

$X = \sum_{i=1}^{k-1} (x_i \wedge y_i + x_k \wedge z_i)$, where $z_i = a_i x_i + b_i y_i$, $1 \leq i \leq k-1$. If $z_i = 0$,

then $\ell(x_i \wedge y_i + x_k \wedge z_i) = 1$. If $z_i \neq 0$, let $a_i \neq 0$, then

$x_i \wedge y_i + x_k \wedge z_i = z_i \wedge (a_i^{-1} y_i - x_k)$, thus $\ell(x_i \wedge y_i + x_k \wedge z_i) \leq 1$.

Hence $\ell(X) \leq k-1$, a contradiction.

REMARK 3. If $X \in \Lambda^2 V$, $\ell(X) = k$ and $X = \sum_{i=1}^k x_i \wedge y_i$, then

$$[X] = \langle x_1, \dots, x_k, y_1, \dots, y_k \rangle.$$

PROOF. Let $U = \langle x_1, \dots, x_k, y_1, \dots, y_k \rangle$; then $[X] \subseteq U$. By Proposition 3, $\dim U = 2k$. Also $X \in \Lambda^2 [X]$; let

$$X = \sum_{i=1}^k x'_i \wedge y'_i, \quad x'_i, y'_i \in [X], \quad 1 \leq i \leq k.$$

Again by Proposition 3, $\dim [X] \geq 2k$. Thus $[X] = U = \langle x_1, \dots, x_k, y_1, \dots, y_k \rangle$.

PROPOSITION 4. If $X, Y \in \Lambda^2 V$, $P(X) = P(Y)$, then $X \sim Y$.

PROOF. Let $X = \sum_{i=1}^k x_i \wedge y_i$, $Y = \sum_{j=1}^s x'_j \wedge y'_j$; then by Remark 3,

$$[X] = \langle x_1, \dots, x_k, y_1, \dots, y_k \rangle \text{ and } [Y] = \langle x'_1, \dots, x'_s, y'_1, \dots, y'_s \rangle.$$

Also by Proposition 3, $P(X) = 2k$, $P(Y) = 2s$. Thus $k = s$. Let T be a linear transformation of V $Tx_i = x'_i$, $Ty_i = y'_i$, $1 \leq i \leq k$; then $C_r(T)X = Y$. Thus $X \sim Y$.

PROPOSITION 5. If $X \in \Lambda^r V$, $\ell(X) = 2$, $X = x_1 \wedge \dots \wedge x_r + y_1 \wedge \dots \wedge y_r$, then $X = \langle x_1, \dots, x_r, y_1, \dots, y_r \rangle$.

PROOF. Let $U = \langle x_1, \dots, x_r, y_1, \dots, y_r \rangle$; then $[X] \subseteq U$. If $[X] \neq U$, then at least one element (say) x_1 is not in $[X]$. Let B be a basis of $[X]$ and extend $\{x\} \cup B$ to a basis of U . Let W be a complement of $\langle x_1 \rangle$ in U , containing $[X]$, i.e., $U = \langle x_1 \rangle \oplus W$, $[X] \subseteq W$. Let $x_i = a_i x_1 + w_i$, $2 \leq i \leq r$ and $y_j = b_j x_1 + w'_j$, $1 \leq j \leq r$, where $w_i, w'_j \in W$. Then $X = X_1 + X_2$, where $X_1 \in \langle x_1 \rangle \wedge (\Lambda^{r-1} W)$ and $X_2 \in \Lambda^r W$, and $\ell(X_i) = 1$, $i = 1, 2$. But $U = \langle x_1 \rangle \oplus W \Rightarrow \Lambda^r U = x_1 \wedge (\Lambda^{r-1} W) \oplus \Lambda^r W$. Also $X \in \Lambda^r [X] \subseteq \Lambda^r W$, hence

$X_1 = X - X_2 \in \Lambda^r W$. Thus $X_1 = 0$ and $X = X_2 \Rightarrow \ell(X) = 1$, a contradiction.

Hence $[X] = U = \langle x_1, \dots, x_r, y_1, \dots, y_r \rangle$.

Note: The above proposition is true also for $\ell(X) = k$.

PROPOSITION 6. If $X, Y \in \Lambda^r V$, $\ell(X) = \ell(Y) = 2$, $P(X) = P(Y)$, then $X \sim Y$.

PROOF. Let $X = x_1 \wedge \dots \wedge x_r + y_1 \wedge \dots \wedge y_r$, $U_1 = \langle x_1, \dots, x_r \rangle$, $U_2 = \langle y_1, \dots, y_r \rangle$, then by Proposition 4, $[X] = U_1 + U_2$. Let $z_1, \dots, z_k$ be a basis of $U_1 \cap U_2$, and extend it to a basis $z_1, \dots, z_k, u_1, \dots, u_s$, where $k + s = r$ of U_1 and to a basis $z_1, \dots, z_k, v_1, \dots, v_s$ of U_2 . Then $P(X) = k + 2s$. Since $x_1, \dots, x_r$ and $z_1, \dots, z_k, u_1, \dots, u_s$ are two bases of U_1 , hence $x_1 \wedge \dots \wedge x_r = az_1 \wedge \dots \wedge z_k \wedge u_1 \wedge \dots \wedge u_s = z_1 \wedge \dots \wedge z_k \wedge \bar{u}_1 \wedge \dots \wedge \bar{u}_s$, where $\bar{u}_1 = au_1$. Similarly $y_1 \wedge \dots \wedge y_r = bz_1 \wedge \dots \wedge z_k \wedge v_1 \wedge \dots \wedge v_s = z_1 \wedge \dots \wedge z_k \wedge \bar{v}_1 \wedge \dots \wedge \bar{v}_s$, where $\bar{v}_1 = bv_1$. Hence $X = z_1 \wedge \dots \wedge z_k \wedge (\bar{u}_1 \wedge u_2 \wedge \dots \wedge u_s + \bar{v}_1 \wedge v_2 \wedge \dots \wedge v_s)$, where $z_1, \dots, z_k, \bar{u}_1, u_2, \dots, u_s, \bar{v}_1, v_2, \dots, v_s$ is a basis of $[X]$. Similarly $Y = z_1' \wedge \dots \wedge z_k' \wedge (\bar{u}_1' \wedge u_2' \wedge \dots \wedge u_s' + \bar{v}_1' \wedge v_2' \wedge \dots \wedge v_s')$, where $z_1', \dots, z_k', \bar{u}_1', u_2', \dots, u_s', \bar{v}_1', v_2', \dots, v_s'$ is a basis of $[Y]$.

Define $T: V \longrightarrow V$, a linear transformation

$Tz_i = z_i'$, $T\bar{u}_1 = \bar{u}_1'$, $Tu_i = u_i'$, $T\bar{v}_1 = \bar{v}_1'$, $Tv_i = v_i'$, for $i = 2, 3, \dots, s$.

Then $C_r(T)X = Y$; hence $X \sim Y$.

REMARK 4. Let $X \in \Lambda^r V$, $\ell(X) = 2$, then $r + 1 \leq \rho(X) \leq 2r$.

PROOF. If $X = x_1 \wedge \dots \wedge x_r + y_1 \wedge \dots \wedge y_r$, then $[X] = \langle x_1, \dots, x_r, y_1, \dots, y_r \rangle = U_1 + U_2$, where $U_1 = \langle x_1, \dots, x_r \rangle$, $U_2 = \langle y_1, \dots, y_r \rangle$. $U_1 \neq U_2$, for otherwise $y_1 \wedge \dots \wedge y_r = ax_1 \wedge \dots \wedge x_r$, where a is a scalar and $\ell(X) = 1$.

$P(X) = 2r - \dim U_1 \cap U_2$. Hence $r+1 \leq P(X) \leq 2r$.

THEOREM 1. Let $E(2, s) = \{X | X \in \Lambda^r V, \ell(X) = 2, P(X) = s\}$, then $E(2, s)$, $s = r+1, r+2, \dots, 2r$ are all the equivalence classes on the set of all vectors of $\Lambda^r V$, of length 2.

PROOF. Follows from Proposition 6 and Remark 4.

PROPOSITION 7. Let $0 \neq X \in \Lambda^r V$ and $x \in V$ such that $x \wedge X = 0$; then $x \in [X]$.

PROOF. Let $x_1, x_2, \dots, x_m$ be a basis of $[X]$. Then $\{\hat{x}_\alpha \mid \alpha \in Q_{r,m}\}$ is a basis of $\Lambda^r[X]$, where $Q_{r,m}$ is a set of all the strictly decreasing sequences of length r on the integers $1, 2, \dots, m$. $\det X = \sum_\alpha a_\alpha \hat{x}_\alpha$; then $x \wedge X = \sum_\alpha a_\alpha x \wedge \hat{x}_\alpha$. If $x \notin [X]$, then $\{x \wedge \hat{x}_\alpha \mid \alpha \in Q_{r,m}\}$ is a part of a basis of $\Lambda^{r+1}\langle x, [X] \rangle$. Thus $x \wedge X = 0 \Rightarrow a_\alpha = 0 \forall \alpha \in Q_{r,m} \Rightarrow X = 0$, a contradiction.

PROPOSITION 8. If $0 \neq X \in \Lambda^r V$ and $x \notin [X]$, then $[x \wedge X] = \langle x \rangle \oplus [X]$.

PROOF. By Proposition 7, $x \wedge X \neq 0$. Again by Proposition 7, since $x \wedge (x \wedge X) = 0$, hence $x \in [x \wedge X]$. Clearly $[x \wedge X] \subseteq \langle x \rangle \oplus [X]$. Let $x, x_1, \dots, x_k$ be a basis of $[x \wedge X]$ and extend it to a basis $x, x_1, \dots, x_k, x_{k+1}, \dots, x_m$ of $\langle x \rangle \oplus [X]$. If $U = \langle x_1, \dots, x_k \rangle$, then $[x \wedge X] = \langle x \rangle \oplus U$, $U \subseteq [X]$. $\Lambda^{r+1}[x \wedge X] = x \wedge (\Lambda^r U) \oplus \Lambda^{r+1}U$. Let $x \wedge X = x \wedge u + v$, where $u \in \Lambda^r U$ and $v \in \Lambda^{r+1}U$. Thus $x \wedge v = 0$. If $v \neq 0$, then by Proposition 7, $x \in [v] \subset U$, a contradiction. Hence $v = 0$ and thus $x \wedge X = x \wedge u$. Then $x \wedge (X - u) = 0$. If $X - u \neq 0$, then by Proposition 7, $x \in [X - u]$. Now $X \in \Lambda^r[X]$ and $u \in \Lambda^r U \subseteq \Lambda^r[X]$; thus $X - u \in \Lambda^r[X]$. Hence $[X - u] \subseteq [X]$. Thus $x \in [X - u] \Rightarrow x \in [X]$, which is a contradiction and therefore $X - u = 0$; i.e., $X = u \in \Lambda^r U$. Hence $[X] \subseteq U$. Also $U \subseteq [X]$, hence $U = [X]$ and $[x \wedge X] = \langle x \rangle \oplus [X]$.

PROPOSITION 9. Suppose $X \in \Lambda^2 V$, $\ell(X) = 2$, x_1, x_2 are linearly independent vectors in $[X]$. Then $\exists y_1, y_2 \in [X]$ and $\lambda \in F \ni X$ has one and only one of the following representations: (i) $X = x_1 \wedge y_1 + x_2 \wedge y_2$,
(ii) $X = \lambda x_1 \wedge x_2 + y_1 \wedge y_2$.

PROOF. $X \in \Lambda^2 V$, $\ell(X) = 2 \Rightarrow P(X) = 4$. Extend x_1, x_2 to a basis x_1, x_2, x_3, x_4 of $[X]$.

Then $X = \sum_{1 \leq i < j \leq 4} a_{ij} x_i \wedge x_j$, $a_{ij} \in F$.

If $a_{34} = 0$, take $y_1 = a_{12}x_2 + a_{13}x_3 + a_{14}x_4$ and $y_2 = a_{23}x_3 + a_{24}x_4$, then

$X = x_1 \wedge y_1 + x_2 \wedge y_2$. If $a_{34} \neq 0$, then

$(-\lambda + a_{12})a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0$ ————— (1) has a solution in F .

Set $Y = (-\lambda + a_{12})x_1 \wedge x_2 + a_{13}x_1 \wedge x_3 + a_{14}x_1 \wedge x_4 + a_{23}x_2 \wedge x_3 + a_{24}x_2 \wedge x_4 + a_{34}x_3 \wedge x_4$.

Then $Y = -\lambda x_1 \wedge x_2 + X$. Because of (1), $\ell(Y) = 1$; also $Y \in \Lambda^2[X]$.

Thus $y_1, y_2 \in [X]$ $Y = y_1 \wedge y_2$. Hence $X = \lambda x_1 \wedge x_2 + y_1 \wedge y_2$.

If $X = x_1 \wedge y_1 + x_2 \wedge y_2$ and also $X = \lambda x_1 \wedge x_2 + z_1 \wedge z_2$ then $x_1 \wedge X = x_1 \wedge x_2 \wedge y_2$

and also $x_1 \wedge X = x_1 \wedge z_1 \wedge z_2$. Thus $0 \neq x_1 \wedge x_2 \wedge y_2 = x_1 \wedge z_1 \wedge z_2$ and hence

$\langle x_1, x_2, y_2 \rangle = \langle x_1, z_1, z_2 \rangle$. Let $z_1 = a_1x_1 + a_2x_2 + a_3y_2$ and

$z_2 = b_1x_1 + b_2x_2 + b_3y_2$.

Then $z_1 \wedge z_2 = (a_1b_2 - a_2b_1)x_1 \wedge x_2 + (a_1b_3 - a_3b_1)x_1 \wedge y_2 + (a_2b_3 - a_3b_2)x_2 \wedge y_2$.

Putting this expression for $z_1 \wedge z_2$ in $X = \lambda x_1 \wedge x_2 + z_1 \wedge z_2$, we get two

different representations of X in the basis of $\Lambda^2[X]$, determined by the basis

x_1, x_2, y_1, y_2 of $[X]$; thus X has precisely one of the two representations.

PROPOSITION 10. If $X, Y \in \Lambda^r V$ are decomposable, then $X+Y$ is decomposable iff $\dim[X] \cap [Y] \geq r-1$.

PROOF. ($\Rightarrow$) Let $X+Y$ be decomposable, and $X+Y = Z$, $\ell(Z) \leq 1$.

Let $X = x_1 \wedge \dots \wedge x_r$, $Y = y_1 \wedge \dots \wedge y_r$, $Z = z_1 \wedge \dots \wedge z_r$. If $[X] = [Z]$, then for

any i , $1 \leq i \leq r$, $z_i \wedge X = z_i \wedge Z = 0$; but then $z_i \wedge Y = 0$, and thus $z_i \in [Y]$

by Proposition 7, and $[Z] = [Y]$. Hence $[X] = [Y]$, i.e., $\dim[X] \cap [Y] = r$.

If $[X] \neq [Z]$, then for some i , $z_i \notin [X]$. But

$z_i \wedge (X+Y) = 0 \Rightarrow z_i \wedge X = -z_i \wedge Y \Rightarrow \langle z_i, [X] \rangle = \langle z_i, [Y] \rangle$. Thus $[X], [Y]$

are r -dimensional subspaces in an $(r+1)$ -dim space $\langle z_i, [X] \rangle$. Hence

$\dim[X] \cap [Y] \geq \dim[X] + \dim[Y] - (r+1) = r-1$. ($\Leftarrow$) If $\dim[X] \cap [Y] \geq r-1$.

Let $u_1, \dots, u_{r-1}$ be l.i. vectors in $[X] \cap [Y]$ and extend these to a basis

$x, u_1, \dots, u_{r-1}$ and a basis $y, u_1, \dots, u_{r-1}$ of $[X]$ and $[Y]$ respectively. Thus

$X = ax \wedge u_1 \wedge \dots \wedge u_{r-1}$, $Y = by \wedge u_1 \wedge \dots \wedge u_{r-1}$ for some a and b .

Hence $X+Y = (ax+by) \wedge u_1 \wedge \dots \wedge u_{r-1}$, i.e., $X+Y$ is decomposable.

THEOREM 2. If $\dim V = 5$, $X \in \Lambda^3 V$, then $\ell(X) \leq 2$.

PROOF. We shall first prove that $\ell(X) \leq 3$. Let x_1, x_2, x_3, x_4, x_5 be a basis of V . Then

$$\begin{aligned} X = \sum_{1 \leq i < j < k \leq 5} a_{ijk} x_i \wedge x_j \wedge x_k &= x_1 \wedge x_2 \wedge (a_{123}x_3 + a_{124}x_4 + a_{125}x_5) \\ &+ x_1 \wedge x_3 \wedge (a_{134}x_4 + a_{135}x_5) + x_2 \wedge x_3 \wedge (a_{234}x_4 + a_{235}x_5) \\ &+ (a_{145}x_1 + a_{245}x_2 + a_{345}x_3)x_4 \wedge x_5. \end{aligned}$$

Let $y_1 = a_{134}x_4 + a_{135}x_5$, $y_2 = a_{234}x_4 + a_{235}x_5$. If y_1, y_2 are l.d., then $\ell(X) \leq 3$. So we assume y_1, y_2 are l.i.; then $\langle y_1, y_2 \rangle = \langle x_4, x_5 \rangle$, and thus

$x_4 \wedge x_5 = \lambda y_1 \wedge y_2$, $\lambda \in F$. Let $a_{124}x_4 + a_{125}x_5 = b_1y_1 + b_2y_2$. Then

$$\begin{aligned} X &= x_1 \wedge x_2 \wedge (a_{123}x_3 + b_1y_1 + b_2y_2) + x_1 \wedge x_3 \wedge y_1 + x_2 \wedge x_3 \wedge y_2 \\ &+ \lambda(a_{145}x_1 + a_{245}x_2 + a_{345}x_3)y_1 \wedge y_2 \\ &= a_{123}x_1 \wedge x_2 \wedge x_3 + (x_1 + a_{345}\lambda y_2) \wedge y_1 \wedge (-b_1x_2 - x_3 + a_{145}\lambda y_2) \\ &+ (b_2x_1 - x_3 - (a_{245} - a_{345}b_1)\lambda y_1) \wedge x_2 \wedge y_2. \end{aligned}$$

Hence $\ell(X) \leq 3$.

Let $X = X_1 + X_2 + X_3$, where X_1, X_2, X_3 are decomposable, $X_1 = x_1 \wedge x_2 \wedge x_3$,

$X_2 = y_1 \wedge y_2 \wedge y_3$, $X_3 = z_1 \wedge z_2 \wedge z_3$. Then $1 \leq \dim[X_1] \cap [X_2] \leq 3$.

CASE 1. $\dim[X_1] \cap [X_2] = 3$. Then $X_2 = \lambda X_1$ for some λ and thus $\ell(X) \leq 2$.

CASE 2. $\dim[X_1] \cap [X_2] = 2$. Let u_1, u_2, v and u_1, u_2, w be bases of $[X_1]$ and $[X_2]$ respectively. Then $X_1 = \lambda u_1 \wedge u_2 \wedge v$ and $X_2 = \lambda u_1 \wedge u_2 \wedge w$. Then $\ell(X) \leq 2$.

CASE 3. $\dim[X_1] \cap [X_2] = 1$. Let u_1, u_2, u_3 and u_1, u_4, u_5 be bases of $[X_1]$ and $[X_2]$ respectively. Then $X_1 = u_1 \wedge u_2 \wedge u_3$, $X_2 = u_1 \wedge u_4 \wedge u_5$; we have assumed the co-effs. to be absorbed with the vectors u_i 's and v_i 's. Then

$X_1 + X_2 = u_1 \wedge Y$, where $Y = u_2 \wedge u_3 + u_4 \wedge u_5$. Also $[X_1] + [X_2] = V$.

Since $\dim\langle u_2, u_3, u_4, u_5 \rangle \cap [X_3] \geq 2$, we can take $X_3 = w_1 \wedge w_2 \wedge w_3$, where $w_1, w_2 \in \langle u_2, u_3, u_4, u_5 \rangle$. By Proposition 9, v_1, v_2 and λ $Y = \lambda w_1 \wedge w_2 + v_1 \wedge v_2$ or $Y = w_1 \wedge v_1 + w_2 \wedge v_2$. If $Y = \lambda w_1 \wedge w_2 + v_1 \wedge v_2$, then $X = u_1 \wedge Y + w_1 \wedge w_2 \wedge w_3$ has length ≤ 2 . If $Y = w_1 \wedge v_1 + w_2 \wedge v_2$, then since u_1, w_1, w_2, v_1, v_2 is also a basis of V , let $w_3 = a_1 u_1 + a_2 w_1 + a_3 w_2 + a_4 v_1 + a_5 v_2$. Then $X = X_1 + X_2 + X_3 = (u_1 - a_4 w_2) \wedge w_1 \wedge v_1 + u_1 \wedge w_2 \wedge v_2 + (a_5 v_2 + a_1 u_1) \wedge w_1 \wedge w_2$ has length ≤ 2 , since $Z = u_1 \wedge w_2 \wedge v_2 + (a_5 v_2 + a_1 u_1) \wedge w_1 \wedge w_2$ and $\dim\langle u_1, w_2, v_2 \rangle \cap \langle a_5 v_2 + a_1 u_1, w_1, w_2 \rangle \geq 2$ implies $\ell(Z) \leq 1$.

REMARK. There exists $X \in \Lambda^3 V$ with $\ell(X) = 2$; for if x_1, x_2, x_3, x_4, x_5 is a basis of V and $X = x_1 \wedge x_2 \wedge x_3 + x_1 \wedge x_4 \wedge x_5$, then $\ell(X) = 2$, by Proposition 10.

REMARK. If $X \in \Lambda^3 V$, $\dim V = 5$, $\ell(X) = 2$, then $P(X) = 5$; for let $X = X_1 + X_2$, where $\ell(X_1) = \ell(X_2) = 1$. Since X is not decomposable, then by Proposition 10, $\dim[X_1] \cap [X_2] < 2$ and hence $\dim[X] > \dim[X_1] + \dim[X_2] - \dim[X_1] \cap [X_2] = 4$, i.e., $P(X) = 5$.

It follows from Proposition 6 that if $X, Y \in \Lambda^3 V$ and $\ell(X) = \ell(Y)$, then $X \sim Y$. Hence all the equivalence classes of $\Lambda^3 V$ are given by

$$S_0 = \{X \mid X \in \Lambda^3 V, \ell(X) = 0\} = \{0\}$$

$$S_1 = \{X \mid X \in \Lambda^3 V, \ell(X) = 1\}$$

$$S_2 = \{X \mid X \in \Lambda^3 V, \ell(X) = 2\}.$$

ACKNOWLEDGMENT. The author is grateful to Professor R. Westwick for his invaluable help in the preparation of this manuscript.

REFERENCES

1. Hodge, W. V. D. and D. Pedoe. Methods of Algebraic Geometry, Vol. 1, University Press, Cambridge, 1953.
2. Lim, M. J. S. L-2 Subspaces of Grassmann Product Spaces, Pacific J. Math. 33 (1970) 167-182.
3. Gurewicz, G. B. Foundations of the Theory of Algebraic Invariants, P. Noordhoff Ltd., Groningen, The Netherlands, 1964.