A FIXED POINT THEOREM FOR A NONLINEAR TYPE CONTRACTION

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ABSTRACT. A well-known result of Boyd and Wong [1] on nonlinear contractions is extended. Several other known results are obtained as special cases.

INTRODUCTION.

In this paper, we extend a well-known result of Boyd and Wong [1] and obtain as consequences several other known results (see [2], [3], [4], [5]).

Throughout this paper, let $(X,d)$ be a complete metric space, $R^+$ the nonnegative reals and $\phi = \phi(t_1,t_2,t_3,t_4,t_5):(R^+)^5 \to R^+$ a function which is (a) continuous from right in each coordinate variable (b) nondecreasing in $t_2$, $t_3$, $t_4$, $t_5$, and satisfies the inequality (c) $\phi(t,s,s,as,bs) < \text{Max}\{t,s\}$ if $\text{Max}\{t,s\} > 0$ where $(a,b) \subseteq (0,1,2)$ with $a + b = 2$. Note that (c) implies that $\phi(t,t,t,t,t) < t$ for any $t > 0$.

2. MAIN RESULTS.

The following is the main result of this paper.

THEOREM 1. Let $f,g:X \to X$ be two commutative mappings such that

(i) $fX \subseteq gX$,

(ii) $g$ is continuous,

(iii) $d(fx,gy) \leq \phi(d(gx,gy)$, $d(fx,gx)$, $d(fy,gy)$, $d(fx,gy)$, $d(fy,gx)$),

for each $x, y \in X$. Then, there exists a unique $u \in X$ with $fu = gu = u$. 
We first prove the following lemma which simplifies the proof of the above theorem.

**Lemma.** Under the conditions of Theorem 1, if there exists a \( v \in X \) such that \( f v = g v \), then there exists a unique \( u \in X \) with \( f u = g u = u \).

**Proof.** We show that for any \( w \in X \)

\[
f(w) = g(w) \implies f(v) = f(w) \tag{2.1}
\]

Suppose \( t = d(fv, fw) > 0 \). Then it follows by (iii) that

\[
t < \phi(t, 0, 0, t, t) < \phi(t, t, t, t, t) < t,
\]

a contradiction. Thus \( f v = f w \). Now, since \( f w = g w \), therefore, \( f(fw) = g(fw) \) and consequently by (2.1)

\[
f(w) = f(fw) = g(fw).
\]

Thus, if we set \( u = f(w) \), then \( f u = g u = u \). The uniqueness of \( u \) now follows from (2.1).

**Proof of Theorem 1.** Let \( x_0 \) be an arbitrary point in \( X \). Construct a sequence \( \{y_n\} \) in \( X \) as follows. Let \( y_0 = fx_0 \). By (i) there exists a \( x_1 \in X \) such that

\[
y_0 = gx_1.
\]

Set \( y_1 = fx_1 \). Thus, if \( y_0, y_1, \ldots y_n \) are obtained with \( y_n = fx_n \), there exists by (i) a \( x_{n+1} \in X \) such that \( y_n = gx_{n+1} \). Let \( y_{n+1} = fx_{n+1} \). Thus, for each \( n \in \mathbb{I} \) (nonnegative integers),

\[
y_n = fx_n = gx_{n+1}. \tag{2.2}
\]

We shall show that \( \{y_n\} \) is a Cauchy sequence in \( X \). For this, let for each

\( n \in \mathbb{I} \), \( d_n = d(y_n, y_{n+1}) \). Then by (i) and (b),

\[
d_{n+1} = d(fy_{n+1}, fy_{n+2}) \leq \phi(d_n, d_n, d_{n+1}, 0, d_n + d_{n+1}). \tag{2.3}
\]

Now, if for some \( n \in \mathbb{I} \), \( d_{n+1} > d_n \), then by (b) and (c)

\[
d_{n+1} \leq \phi(d_n, d_{n+1}, d_{n+1}, 0, 2d_{n+1}) < d_{n+1}.
\]
a contradiction. Thus for each \( n \in I \), \( d_{n+1} \leq d_n \), that is \( \{d_n\} \) is a nonincreasing sequence of nonnegative reals and consequently there exists a \( d \in \mathbb{R}^+ \) such that 
\[ \{d_n\} \to d. \]
Clearly \( d = 0 \), for otherwise by (2.3) and (c),
\[ d < \phi(d,d,d,0,2d) < d, \]
a contradiction. Thus,
\[ d_n \to 0. \quad (2.4) \]
Suppose, now that \( \{y_n\} \) is not a Cauchy sequence. Then there exists a \( E > 0 \) such that for each \( k \in I \), there exist integers \( n(k), m(k) \) with \( k \leq n(k) < m(k) \) satisfying
\[ E_k = d(y_{n(k)}, y_{m(k)}) > E. \]
Let \( m(k) \) be the least integer greater than \( n(k) \) such (2.4) holds. This implies that for each \( k \in I \),
\[ E < E_k \leq d(y_{n(k)}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{m(k)}) \leq E + d_k. \quad (2.5) \]
Hence, it follows by (2.4) that as \( k \to \infty \), \( E_k \to E. \)
However, for each \( k \in I \),
\[ E_k \leq d_n(k) + d(fx_{n(k)} + f_{m(k)+1}) + d_m(k), \]
\[ \leq 2d_k + \phi(E_k, d_k, d_k, E_k, d_k, E_k, d_k), \]
Therefore, as \( k \to \infty \),
\[ E \leq \phi(E, 0, 0, E, E) < E, \]
contradicting the existence of \( E > 0 \). Thus, \( \{y_n\} \) is a Cauchy sequence in \( X \).
Consequently, there is a \( v \in X \) such that \( \{y_n\} \to v \), that is
\[ fx_n = gx_{n+1} \to v. \quad (2.6) \]
We show that for this \( v \),
\[ a = d(fv, gv) = 0. \]
Suppose $\alpha > 0$. Now by (ii) and (2.6) we have,
\[ f^g x_n = g^f x_n \rightarrow g^v \text{ and } g^{2x_n} \rightarrow g^v. \]

Also, it follows by (b) and (iii) that,
\[ d(f(g^x_n), f^v) \leq d(g^{2x_n}, g^v), \quad d(f^g x_n, g^{2x_n}), \quad \alpha, \quad d(f^g x_n, g^v), \quad \alpha + d(g^v, g^{2x_n}). \]

Therefore, as $n \rightarrow \infty$, the above inequality yields that
\[ \alpha = d(g^v, f^v) \leq \phi(0, 0, \alpha, \alpha) < \alpha, \]
a contradiction. Thus $f^v = g^v$ and hence by the above lemma, there is a unique $u \in X$ satisfying $f^u = g^u = u$.

In the special case when $g$ is taken to be the identity map of $x$ in Theorem i, we have

COROLLARY i. Let $f: X \rightarrow X$ satisfy either of the following conditions: for all $x, y \in X$,

(A). \[ d(fx, fy) \leq \phi(d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)). \]

(B). \[ d(fx, fy) \leq \alpha(d(x, fx) + d(y, fy)) + \beta(d(x, fy) + d(y, fx)) + \psi(d(x, y)) \]

where $\alpha > 0$, $\beta > 0$ and $\psi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a right continuous function satisfying
\[ \psi(t) < (1-2\alpha-2\beta)t \text{ if } t > 0. \]

Then $f$ has a unique fixed point in $X$.

PROOF. The conclusion is an obvious consequence of Theorem 1 if (A) holds. In case of condition (B), let $\phi: (\mathbb{R}^+)^5 \rightarrow \mathbb{R}^+$ be defined by
\[ \phi(t_1, t_2, t_3, t_4, t_5) = \psi(t_1) + \alpha(t_2 + t_3) + \beta(t_4 + t_5). \]
then $\phi$ satisfies conditions (a), (b) and (c). Thus the conclusion again follows by Theorem 1.

It may be remarked that if $\alpha = \beta = 0$ in (B) then Corollary 1 yields a well-known result of Boyd and Wong [1]. If $\psi(t) = at$, then Corollary 1 yields certain results of Hardy and Rogers [2], Kannan [3], Reich [4], Sehgal [5]. All these results are special cases of Theorem 1.
REFERENCES


KEY WORDS AND PHRASES. Nonlinear type contraction, Fixed point theorems.
