ABSTRACT. In this paper we obtain the general solution of scalar, first-order differential equations. The method is variation of parameters with asymptotic series and the theory of partial differential equations.

The result gives us a form like a differential quotient requiring only that a limit be taken. Like the familiar expression for the solution of linear, first order, ordinary equations, it is the same in all cases.


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1. INTRODUCTION.

We present a unified treatment for the general scalar, first-order, ordinary differential equation

\[ y' = G(x,y), \quad G \in C^1. \]
Particular examples are linear equations, Riccati equations and Abel equations.

2. PRELIMINARIES.

We begin with the differential system

\[
\begin{align*}
V_1' &= f(V_1, V_2) = -V_1 V_2 \\
V_2' &= h(V_1, V_2) = V_1 - V_2 \\
V \neq 0
\end{align*}
\]

with general solution \( V_1 = V_1(x, c_1, c_2), \quad V_2 = V_2(x, c_1, c_2) \). Here \( c_1, c_2 \) are arbitrary constants.

Now let \( x = x(t) \). Then we get

\[
\begin{align*}
\dot{V}_1 &= U_1 \dot{x} \\
V_2 &= U_2 \dot{x} \\
U_1 &= f(V_1, V_2), \quad U_2 = h(V_1, V_2) \\
V_1 &\neq 0
\end{align*}
\]

We are now ready to present the algebraic system referred to in the title.

3. THE CAUCHY-KOWALEWSKI SYSTEM.

Let \( w_1 = w_1(t, \varepsilon), \quad w_2 = w_2(t, \varepsilon) \) be two functions of \( t \) and \( \varepsilon \) (at present unknown).

The functions \( V_1, V_2 \) have been given by (2.1). Finally two more unknown functions \( K(w_1, w_2, t, \varepsilon) \) and \( L(w_1, w_2, t, \varepsilon) \) will be defined by partial differential equations later. They will contain another variable, \( \lambda \). It will be possible to substitute an arbitrary \( G(w_1, t) \) for \( \lambda \) to solve specific equations.
DEFINITION. The system of algebraic equations

\[
\begin{cases}
(a) \quad w_1 - K(w_1, w_2, t, \varepsilon)V_1 = 0 \\
(b) \quad w_2 - L(w_1, w_2, t, \varepsilon) - V_2 = 0 \\
(c) \quad x = w_1 + tw_2 \\
\end{cases}
\]

is called the Cauchy-Kowalewski system, for a specific \( G(w_1, t) \). Using \( \lambda \) we will get a universal system.

Under suitable conditions on the functions \( K \) and \( L \), we can solve it for \( w_1 = w_1(t, \varepsilon) \) and \( w_2 = w_2(t, \varepsilon) \). We proceed by defining these functions as solutions of appropriate partial differential equations. We will derive these functions \( L(w_1, w_2, t, \varepsilon, \lambda) \) and \( K(w_1, w_2, t, \varepsilon, \lambda) \) and regard them as fixed like universal constants.

4. THE FIRST FUNCTION \( K \) IN THE CAUCHY-KOWALEWSKI SYSTEM.

We differentiate 3(a-b) with respect to \( t \) to get expressions for \( \dot{w}_1, \dot{w}_2 \).

Denoting the expression for \( \dot{w}_1 \) by \( R \) we get

\[
\dot{w}_1 = R
\]

To simplify notation, let \( K = \alpha \) in (4.1) and get

\[
\dot{w}_1 = R = \frac{A_1L_2 + A_2L_3 + A_3}{-A_2L_1 + A_4L_2 + A_5}
\] (4.1a)

Some of the \( A_i \), \( i = 1, \ldots, 5 \) are given explicitly later. These are not partial derivatives. By contrast,

\[ L_1 = \frac{3L}{3w_1} \text{ etc.} \]

Now let \( z = L - w_2^2 \) and note that from (2.1), 3(a-b) we have \( f = \frac{w_1}{\alpha}z \),

\( h = \frac{w_1}{\alpha} + z \) in the new notation.
The following equation is of fundamental importance. We arbitrarily set

\[ A_2 = 2w_1w_2a_1 - w_1tha_1 + w_1a_3 + tfa^2 = \varepsilon \quad (4.2) \]

where \( K = -a = a(L, w_1, w_2, t, \varepsilon) \) and \( a_1 = \frac{3a}{aL} \), etc., for real \( \varepsilon > 0 \).

By the Cauchy-Kowalewski theorem [See e.g. (2.1)] let \( a_o = a_o(L, w_1, w_2, t, \varepsilon) \) be an analytic solution of (4.2). Further, we will write

\[ A_i = A_i(a_o), \quad i = 1, 2, 3, 4, 5. \]

Let \( a_o = \sum_{n=0}^{\infty} c_n \varepsilon^n \) where \( c_n = c_n(L, w_1, w_2, t) \) are analytic. Before imposing conditions on \( c_o \) we give the following definitions.

**DEFINITION.** \( \Delta = \sum_{i=1}^{\infty} a_{i-1}a_i \).

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Two more of the \( A_i \) will now be given explicitly.

\[ \bar{A}_1 = \frac{w_1}{a_o} + z(w_1\alpha_0 - w_1w_2\alpha_2 + w_2\alpha_0) \]

\[ \bar{A}_4 = w_1\alpha_0 + z(a_0^2 - \alpha_0 - w_1h\alpha_0) \]

**DEFINITION.** \( \Delta = \sum_{i=1}^{\infty} a_{i-1}a_i \).

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The conditions on \( c_o \) can be stated now as follows:

1. \( c_o \not\equiv 0 \),
2. \( S_1(L, w_1, w_2, t) \not\equiv 0 \),
3. \( \Delta \not\equiv 0 \).

Substituting \( a_o = \sum_{n=0}^{\infty} c_n \varepsilon^n \) in (4.2) we get

\[ 2w_1w_2c_{o1} - w_1t(w_1c_{o1} + z)c_{o1} + w_1c_{o3} + tw_1zc_{o} = 0 \quad (4.3) \]
of which some solutions are given

$$H[β(c_o,z,w_1),w_2 + \frac{1}{t} P(c_o,w_1,β(c_o,z,w_1))] = \text{constant} \tag{4.3a}$$

where

(1) $H$ is arbitrary

(2) $β$ satisfies the partial differential equation

$$w_1 β_1 + \frac{w_1}{c_o} + z)β_2 = 0$$

$$w_1 β_3 + c_o β_1 \neq 0$$

(3) $P$ is defined as follows: first solve $β(c_o,w_1,z) = a$ for $z = Q(c_o,w_1,a)$. Then set

$$P = \int \frac{d c_o}{c_o Q(c_o,w_1,a)} .$$

THEOREM 1. The function $H$ can be chosen analytic in (4.3a) so that conditions (2.1), (2.2), (3.1) hold for $c_o$.

PROOF. Let $γ = w_2 + \frac{1}{t} P$ and then (4.3a) becomes $H(β,γ) = \text{constant}$. The partial derivatives of $c_o$ are computed from (4.3a) and from them we see that $H_γ \neq 0$ implies that $\frac{∂c_o}{∂γ} \neq 0$, so condition (2.1) holds. Further, $Δ = L A_1 = 0$ implies $(\frac{P}{c_o} + w_2)H_γ = 0$. So $H_γ \neq 0$ implies $Δ \neq 0$. Thus (2.1), (2.2) hold if merely $H_γ \neq 0$. Now $S_1 = 0$ implies that $t w_2(w_1 β_3 + c_o β_1)H_β + H_γ = 0$. Since $w_1 β_3 + c_o β_1 \neq 0$, we can choose $H$ so that $S_1 \neq 0$. This completes the proof.

Summarizing the results of this section, $K = α = α_0$ can be defined as the solution of (4.2) where $H$ is analytic, $c_o \neq 0$, $S_1 \neq 0$, and $Δ \neq 0$. To solve (3.1) however, we must define $L$.

5. SOLUTION OF THE CAUCHY-KOWALEWSKI SYSTEM.

To solve the system (3.1), we must now define the function $L(w_1,w_2,t,ε)$. 
Setting \( \dot{\omega} = G, \alpha = \alpha_0 \) and \( A_2 = \varepsilon \), (4.2) in (4.1a) suggests defining \( L \) by

\[
\varepsilon GL_1 + (A_1 - GA_4)L_2 + eL_3 = GA_5 - A_3.
\]

\( L_1 = \frac{\partial L}{\partial \omega_1} \), etc. This does not seem to be feasible. Instead, letting \( \varepsilon \) tend to zero leads to

\[
L_2 = \frac{\partial L}{\partial \omega_2} = \frac{GA_5 - A_3}{A_1 - GA_4}\tag{5.1}
\]

This will be used to define \( L \).

Let \( \lambda \) be a new variable and consider

\[
L_2 = \frac{\lambda A_5 - A_3}{A_1 - \lambda A_4}\tag{5.2}
\]

Note that the right side of (5.2) is analytic where \( \omega_1 \neq 0 \) and \( A_1 - \lambda A_4 \neq 0 \). So let \( L = \overline{L}(\omega_1, \omega_2, t, \varepsilon, \lambda) = P_1(\omega_2) + P_2(\omega_1, \omega_2, t, \varepsilon, \lambda) \) be an analytic solution on (5.2) and assume that none of the expressions \( \Delta, S_1, \zeta \) vanish when \( L = P_1(\omega_2) \).

Now since the value of \( \frac{\partial}{\partial \omega_2} \overline{L}(\omega_1, \omega_2, t, \varepsilon, \lambda) \) for \( \lambda = G(\omega_1, t) \) is the same as \( \frac{\partial}{\partial \omega_2} \overline{L}(\omega_1, \omega_2, t, \varepsilon, G(\omega_1, t)) \) we see that \( \overline{L}(\omega_1, \omega_2, t, \varepsilon, \lambda) = \overline{L}(\omega_1, \omega_2, t, \varepsilon, G(\omega_1, t)) \) is a solution of (5.1) for any \( G \). Moreover \( \overline{L} \in \mathcal{C}^\infty \) since \( G \) is continuous and \( \overline{L} \) is analytic. Let \( K_G = a_0(\overline{L}, \omega_1, \omega_2, t) \) and \( L = \overline{L} \).

We now prove the solvability near suitable points of the Cauchy-Kowalewski system. The variable \( \lambda \) gives our functions the universal character referred to previously.

**Lemma I.** Let \( (a, b, c) \) be such that \( S_1(P_1(b)a, b, c) \neq 0 \). Then, for small \( t \), the Jacobian of (3.1) is nonzero at \( (a, b, c, \varepsilon) \).
PROOF. If the Jacobian of (3.1) = 0, then

\[-A_2 L_1 + A_4 L_2 + A_5 = 0\]  \hspace{1cm} (5.3)

The subsidiary equations of (5.3) are:

\[\frac{dw_1}{dL} = \frac{dw_2}{dL} = \frac{dL}{A_2 A_4 - A_5}, \text{ so that } \frac{dL}{dw_2} = \frac{-A_5}{A_4}\]

But from (5.1), \[\frac{dL}{dw_2} = \frac{GA_5 - A_3}{A_1 - GA_4}\]

Thus \[\frac{A_1 A_5}{A_3} = \frac{A_1 A_5}{A_3 A_4} = 0\].

But \[\frac{A_1 A_5}{A_3 A_4} = (w_1 \alpha_0^4 - w_1 w_2 \alpha_0^2 + w_2 \alpha_0) (\frac{w_1}{\alpha_0} + z)\]. So

\[L \frac{(w_1 \alpha_0^4 - w_1 w_2 \alpha_0^2 + w_2 \alpha_0) (\frac{w_1}{\alpha_0} + z)}{\varepsilon} = 0\]. However

\[L \frac{(w_1 \alpha_0^4 - w_1 w_2 \alpha_0^2 + w_2 \alpha_0) (\frac{w_1}{\alpha_0} + z)}{\varepsilon} = S_1 (P_1 (w_2), w_1, w_2, t) \neq 0\] and the proof is complete.

We next consider continuity in order to apply the implicit function theorem to (3.1). We first observe that \[L \frac{A_1}{\varepsilon} \neq 0\]. If \[L \frac{A_4}{\varepsilon} = 0\], then

\[L \frac{(A_1 - GA_4)}{\varepsilon} \neq 0\].

Now consider the case where \[L \frac{A_4}{\varepsilon} \neq 0\], but \[L \frac{(A_1 - GA_4)}{\varepsilon} = 0\].

**LEMMA II.** There is at most one function \(G\) such that \[L \frac{(A_1 - GA_4)}{\varepsilon} = 0\].

**PROOF.** Let \[L (w_1, w_2, t, \varepsilon) = L (w_1, w_2, t, G(w_1, t)) = P_1 (w_2) + \varepsilon P_2 (w_1, w_2, t, \varepsilon, G(w_1, t))\]. So it and its partials with respect to \(w_1, w_2, t\) do not contain \(G\) as \(\varepsilon \to 0\). Since

\[\alpha_0 = \sum_{n=0}^{\infty} c_n (L, w_1, w_2, t) = c_0 (L, w_1, w_2, t) + c_1 (L, w_1, w_2, t) + c_2 (L, w_1, w_2, t) \varepsilon^2 + \ldots\]
the same holds for it.

Thus \[ \lim_{\varepsilon \to 0} \frac{A_1}{A_4} \] and \[ \lim_{\varepsilon \to 0} \frac{A_4}{A_1} \] are independent of \( G \).

So \[ G = \lim_{\varepsilon \to 0} \frac{A_1}{A_4} \]. This completes the proof.

In the sequel, we ignore this possible exception and assume that
\[ \lim_{\varepsilon \to 0} \left( \frac{A_1}{A_4} - GA_4 \right) \neq 0 \] for any \( G \).

**LEMMA III.** If \((a,b,c)\) is such that \( S_2(P_1(b),a,b,c) \neq 0 \), there is an \( \varepsilon > 0 \) such that the left sides of (3.1) are \( C^1 \) at \((a,b,c,\varepsilon)\).

**PROOF.** Based on analytic properties of \( V_1,V_2,L,K,G \) and the nonvanishing of \( S_2 \), we will not give details.

Choosing constant values for \( w_1,w_2 \) in (3.1), we can get \( c_1(\varepsilon),c_2(\varepsilon) \) so that left sides vanishes and apply the implicit function theorem to (3.1). Then we solve for \( w_1(t,\varepsilon) \) and \( w_2(t,\varepsilon) \). Here \( c_1,c_2 \) come from equation (2.1) of section 2.

6. **THE PRINCIPAL DIFFERENTIAL EQUATION.**

We now consider the differential equation

\[ \frac{dy}{dx} = y' = g(x,y) \quad (6.1) \]

**DEFINITION.** \( W_1(t) = \lim_{\varepsilon \to 0} W_1(t,\varepsilon) \).

It will be shown that \( W_1(t) \) satisfies (6.1). Of course we change \( y,x \) to \( W_1,t \) respectively.

We begin this process with

**THEOREM II.** Let \( S_1 \neq 0 \) at \((\overline{W_1,\overline{W}_2,\overline{E}})\). Then \[ \lim_{\varepsilon \to 0} \frac{\partial}{\partial t} W_1(t,\varepsilon) + G(\overline{W_1,\overline{E}}) \] as \( \varepsilon \to 0 \).

**PROOF.** \[ \frac{d}{dt} \overline{L} = P_1(w_2) + \varepsilon P_2(w_1,w_2,t,G(w_1,t)) \] so that \[ \frac{3^L}{3w_1} \to 0 \] as \( \varepsilon \to 0 \) and also \( \frac{\partial L}{\partial t} \to 0 \) as \( \varepsilon \to 0 \). Thus \[ L_1, L_3 \to 0 \] as \( \varepsilon \to 0 \).
Now (5.1) \( \bar{A}_1L_2 + \bar{A}_3 = G(\bar{A}_4L_2 + \bar{A}_5) \).

Also \( \bar{A}_4L_2 + \bar{A}_5 = \frac{\bar{A}_1A_5 - \bar{A}_3A_4}{\bar{A}_1 - GA_4} = \frac{S_1\varepsilon}{\bar{A}_1 - GA_4} \).

Thus \( R = \frac{\bar{A}_1L_2 + \bar{A}_2L_3 + \bar{A}_3}{\bar{A}_2L_1 + \bar{A}_4L_2 + \bar{A}_5} = \frac{\varepsilon L_3 + \bar{A}_1L_2 + \bar{A}_3}{-\varepsilon L_1 + \bar{A}_4L_2 + \bar{A}_5} \).

So \( \dot{w}_1 = \frac{\varepsilon L_3 + G(\bar{A}_4L_2 + \bar{A}_5)}{-\varepsilon L_1 + (\bar{A}_4L_2 + \bar{A}_5)} = \frac{(\bar{A}_1 - GA_4)L_3 + GS_1}{-(\bar{A}_1 - GA_4)L_1 + S_1} \).

Therefore \( \dot{w}_1 + \frac{GS_1}{S_1} \) as \( \varepsilon \to 0 \) and \( S_1 \neq 0 \). This completes the proof.

By the last theorem, \( L \frac{d}{\varepsilon \to 0} w_1(t,\varepsilon) = G(w_1(t,\varepsilon),t) = G(Lw_1(t,\varepsilon),t) = G(w_1(t,\varepsilon),t) = G(W_1(t),t) \).

But also it is true [2: P.461] that

\[
L \frac{d}{\varepsilon \to 0} w_1(t,\varepsilon) = \frac{d}{\varepsilon \to 0} (Lw_1(t,\varepsilon)) = \dot{w}_1(t).
\]

So \( \dot{w}_1(t) = G(W_1(t),t) \) \hspace{1cm} (6.2)

7. PARTICULAR AND GENERAL SOLUTIONS OF \( y' = G(x,y) \).

7(a) PARTICULAR SOLUTIONS. Let \( J(w_1,t) \in C^1 \),

\[
L^*(w_1,w_2,t) = L(w_1,w_2,t,\varepsilon,J(w_1,t)) \quad \text{and} \quad \alpha^*(w_1,w_2,t) = \alpha_0(L^*,w_1,w_2,t).
\]

Let \( Q \) be the set of points in \((w_1,w_2,t)\)-space where

(1) \( w_1 \neq 0 \quad (2) \ c_0 \neq 0 \quad (3) \ S_1 \neq 0 \quad (4) \ S_2 \neq 0 \).

Let \( \bar{Q} \) be the projection of \( Q \) on the \((w_1,t)\) plane.

The Universal Cauchy-Kowalewski System

DEFINITION. \( \bar{\alpha}(w_1,w_2,t,\varepsilon,\lambda) = \alpha_0(L,w_1,w_2,t) \).

DEFINITION. \( F_1 \equiv w_1 - \bar{\alpha}V_1(w_1 + tw_2,c_1,c_2) \).
DEFINITION. $F_2 = \omega_2^2 - \overline{L}(\omega_1, \omega_2, t, \varepsilon, \lambda) - V_2(\omega_1 + t\omega_2, c_1, c_2)$.

DEFINITION. $F_3 = \overline{A}_1 - J(\omega_1, t)\overline{A}_4$ with $\lambda$ replaced by $J(\omega_1, t)$.

DEFINITION. The system $\begin{cases} F_1 = 0 \\ F_2 = 0 \\ F_3 \neq 0 \end{cases}$ is also called the Universal Cauchy-Kowalewski System.

We refer to it in the following theorem.

**THEOREM III.** Let $P \in \overline{Q}$. There is a region in which the solution through $P$ of $\dot{\omega}_1 = J(\omega_1, t)$ is determined as follows:

1. In $F_1, F_2$ replace $\lambda$ by $J(\omega_1, t)$ and $c_1, c_2$ by suitable functions of $\varepsilon$.
2. Equate the results in (2.1) to zero.
3. Solve the resulting system for $\omega_1(t, \varepsilon)$ and $\omega_2(t, \varepsilon)$.
4. Take the limit of $\omega_1(t, \varepsilon)$ as $\varepsilon \rightarrow 0$.

**PROOF.** Let $P = (a, t_0)$, $P \in \overline{Q}$. Since $c_0(\overline{P}(b), a, b, t_0) \neq 0$, there is an $\varepsilon$ such that $\alpha^*(a, b, t_0, \varepsilon) \neq 0$. Let $(\overline{V}_1, \overline{V}_2)$ be a solution of (2.1) such that

\[
\begin{align*}
\overline{V}_1(a + t_0 b) &= \frac{a}{\alpha^*(a, b, t_0, \varepsilon)} \\
\overline{V}_2(a + t_0 b) &= b^2 - L^*(a, b, t_0, \varepsilon).
\end{align*}
\]

Solve the system:

\[
\begin{align*}
(1) \quad \overline{V}_1(a + t_0 b, c_1, c_2) - \frac{a}{\alpha^*(a, b, t_0, \varepsilon)} &= 0 \\
(2) \quad \overline{V}_2(a + t_0 b, c_1, c_2) - b^2 - L^*(a, b, t_0, \varepsilon) &= 0
\end{align*}
\]

to get suitable $c_1 = c_1(\varepsilon)$, $c_2 = c_2(\varepsilon)$.

Since $S_1 \neq 0$ our system has nonzero Jacobian. We solve for $\omega_1(t, \varepsilon)$ and get the result.
7(b) GENERAL SOLUTIONS. Alternatively, eliminating $w_2$ from the Universal Cauchy-Kowalewski System we get

$$X(w_1, t, e, \lambda, c_1, c_2) = 0$$

(7.1)

where $c_1, c_2$ are constants.

The general solution of a specific equation is obtained as follows:

(1) Replace $\lambda$ by $G(w_1, t)$ in (7.1).

(2) Take the limit as $e \to 0$ of the result.

$X$ is derived from $L$ and $K$ and is like the familiar differential quotient in generality.

**REFERENCES**

