ON THE MEIJER TRANSFORMATION

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ABSTRACT. Recently [8], an operational calculus for the operator $B_\mu = t^{-\mu} D t^{1+\mu} D$ with $-1 < \mu < \infty$ was developed via the algebraic approach [4], [13], [15]. This paper gives the integral transform version. In particular, a differentiation theorem and a convolution theorem are proved.

1. INTRODUCTION.

Ditkin [4], and later with Prudnikov [6], developed an operational calculus for the operator $\frac{d}{dt} t \frac{d}{dt}$ similar to the algebraic approach of Mikusinski [15]. Meller [13], [14] generalized Ditkin's calculus to operators $B_\alpha = t^{-\alpha} D t^{1+\alpha} D$ with $-1 < \alpha < 1$. Krätzel [9], [10], [11], [12] gave an integral transform version to Meller's calculus and also generalized the calculus to operators containing n^{th} order derivatives. He developed a theory of integral transforms of the form

$$\mathcal{L}_v^{(n)}\{f\}(s) = \int_0^\infty w_v^{(n)}(n(st)^{1/n})f(t)dt,$$

where $n = 1, 2, \dots$, $\operatorname{Re}(v) > \frac{1}{n} - 1$, and

$$w_v^{(n)}(z) = \frac{(2\pi)^{\frac{n-1}{2}} \sqrt{n} \left(\frac{z}{n}\right)^{nv}}{\Gamma(v+1-1/n)} \int_1^\infty (y^n-1)^{v-\frac{1}{n}} \exp(-zy) dy.$$

Here, $\mathcal{L}_v^{(1)}$ is the Laplace transform and $\mathcal{L}_v^{(2)}$ is the Meijer transform of the form

$$\mathcal{L}_v^{(2)}\{f\}(s) = 2 \int_0^\infty (st)^{v/2} K_v(2\sqrt{st}) f(t) dt, \quad (1)$$

where $K_v(z)$ is the MacDonald function of order v . Dimovski [1], [2], [3] developed an operational calculus for the operator

$$B = t^{\alpha_0} \frac{d}{dt} t^{\alpha_1} \frac{d}{dt} \dots t^{\alpha_{n-1}} \frac{d}{dt} t^{\alpha_n},$$

using an integral transform that for $n = 2$ reduces to the Meijer transform of the form

$$\tilde{k}_v\{f\}(s) = 2s^{-v} \int_0^\infty (st)^{v/2} K_v(2\sqrt{st}) f(t) dt. \quad (2)$$

In [8], Koh reconsidered Meller's operator $B_\mu = t^{-\mu} \frac{d}{dt} t^{1+\mu} \frac{d}{dt}$ but with $\mu \in (-1, \infty)$. Following Mikusinski, Ditkin, et. al., he constructed an operational calculus through the field extension of a commutative convolution ring without zero divisors. His calculus reduces to Ditkin's when $\mu = 0$ and Meller's when $\mu \in (-1, 1)$.

In this paper, we give an integral transform analogue of [8] via the Meijer transform of the form

$$k_{\mu}\{f\}(p) = \frac{2p}{\Gamma(\mu+1)} \int_0^{\infty} (pt)^{\mu/2} K_{\mu}(2\sqrt{pt}) f(t) dt \quad (3)$$

for $\operatorname{Re}(\mu) > -1$. In particular, we prove a differentiation theorem and a convolution theorem. The presence of a factor $\frac{2p}{\Gamma(\mu+1)}$ in (3), as opposed to those in (1) and (2), is essential in our convolution theorem.

2. THE MAIN THEOREMS.

We will define the convolution, $*$, of two functions, f , g by

$$f * g = \frac{1}{\Gamma(\mu+1)} D^{\mu+1} \int_0^t \xi^{\mu} (t-\xi)^{\mu} \int_0^1 f(x\xi) g[(1-x)(t-\xi)] dx d\xi, \quad (4)$$

see Koh [8], where D^{λ} is the Riemann-Liouville derivative of order λ , see Ross [17]. This convolution exists if, for example, f and g are in $C^{\infty}[0, \infty)$, the space of infinitely differentiable complex functions on $[0, \infty)$.

The following properties of K_{μ} will be used:

$$K_{\mu}(2\sqrt{pt}) = \frac{1}{2}(t/p)^{\mu/2} \int_0^{\infty} x^{-\mu-1} \exp(-px - \frac{t}{x}) dx \quad (5.1)$$

$$= \frac{1}{2}(t/p)^{-\mu/2} \int_0^{\infty} x^{\mu-1} \exp(-px - \frac{t}{x}) dx, \quad (5.2)$$

$\operatorname{Re}(\mu) > -1$, $\operatorname{Re}(p) > 0$, $\operatorname{Re}(t) > 0$.

$$2(pt)^{\mu} K_{\mu}(2\sqrt{pt}) \sim \left\{ \begin{array}{l} -\ln t + O(1), \mu = 0 \\ \Gamma(\mu) + O[t^{\min(1, \mu)}], \mu > 0 \\ -\frac{\Gamma(1-\mu)}{\mu} (pt)^{\mu+O(1)}, -1 < \mu < 0 \end{array} \right\}, t \rightarrow 0 \quad (6.1)$$

$$\sim \sqrt{\frac{\pi}{2}} t^{\frac{\mu}{2} - \frac{1}{4}} e^{-2\sqrt{pt}} \{1 + O(|t|^{-\frac{1}{2}})\}, t \rightarrow \infty. \quad (6.2)$$

$$\frac{d}{dt} \{(pt)^{\pm\mu/2} K_{\mu}(2\sqrt{pt})\} = -p(pt)^{\pm\mu-\frac{1}{2}} K_{\mu\pm 1}(2\sqrt{pt}). \quad (7)$$

In order that the Meijer transform (3) converges, it is sufficient for $f(t)$ to be locally Lebesgue integrable on $(0, \infty)$ and $|f(t)| < Ce^{2\gamma\sqrt{t}}$ ($t \rightarrow \infty$) for $\mu > 0$ and for $f(t)$ to remain bounded in the neighbourhood of the origin for $-1 < \mu \leq 0$. The integral then converges absolutely within the parabolic region $\operatorname{Re}\sqrt{p} > \gamma$. This is clear from the asymptotic behaviors (6.1) and (6.2). Indeed,

$$\begin{aligned} \left| \int_0^{\infty} f(t) (pt)^{\frac{\mu}{2}} K_{\mu}(2\sqrt{pt}) dt \right| &\leq \int_0^{\infty} |f(t)| (pt)^{\frac{\mu}{2}} K_{\mu}(2\sqrt{pt}) dt \\ &\leq \sup_{0 < t < \varepsilon} |f(t)| \int_0^{\varepsilon} (pt)^{\frac{\mu}{2}} K_{\mu}(2\sqrt{pt}) dt + \int_{\varepsilon}^T |f(t)| (pt)^{\frac{\mu}{2}} K_{\mu}(2\sqrt{pt}) dt \\ &\quad + \int_T^{\infty} |f(t)| (pt)^{\frac{\mu}{2}} K_{\mu}(2\sqrt{pt}) dt, \text{ for some } 0 < \varepsilon < T < \infty. \end{aligned} \quad (8)$$

The first integral on the right hand side of (8) exists because of (6.1); the second exists because of the local integrability of $f(t)$ and the continuity of $(pt)^{\frac{\mu}{2}} K_{\mu}(2\sqrt{pt})$; and the last

integral exists because of (6.2) provided $\operatorname{Re}\sqrt{p} > \gamma$. We state this result in

THEOREM 1. If $f(t) \in L_{\text{loc}}(0, \infty)$, if there are constants C and γ such that $|f(t)| < Ce^{2\gamma\sqrt{t}}$ as $t \rightarrow \infty$, and if $\lim_{t \rightarrow 0^+} f(t) = f(0^+) < \infty$, then (3) converges absolutely in $\operatorname{Re}\sqrt{p} > \gamma$ for all $\mu \in (-1, \infty)$. Furthermore, the integral (3) as a function of p is analytic in the region of convergence.

The proof of the analyticity is standard and is omitted. When a function $f(t)$ satisfies the hypothesis of theorem 1, we shall write, for brevity, $f \in \text{HypI}$. Clearly, if a function f has continuous derivative on $[0, \infty)$ and $f' \in \text{HypI}$, then $f \in \text{HypI}$.

THEOREM 2. If $f \in C^2[0, \infty)$ and $f' \in \text{HypI}$, then

$$k_{\mu}(B_{\mu}f) = p(k_{\mu}f) - pf(0^+).$$

PROOF.

$$\begin{aligned} k_{\mu}(B_{\mu}f) &= \frac{2p^{\frac{\mu}{2}+1}}{\Gamma(\mu+1)} \int_0^{\infty} [t^{-\mu} \frac{d}{dt} t^{1+\mu} \frac{d}{dt} f(t)] t^{\frac{\mu}{2}} K_{\mu}(2\sqrt{pt}) dt \\ &= \frac{2p^{\frac{\mu}{2}+1}}{\Gamma(\mu+1)} \{ t^{-\frac{\mu}{2}} K_{\mu}(2\sqrt{pt}) t^{\mu+1} \frac{df}{dt} \Big|_0^{\infty} - \int_0^{\infty} (t^{\mu+1} \frac{df}{dt}) \frac{d}{dt} (t^{-\frac{\mu}{2}} K_{\mu}(2\sqrt{pt}) dt) \}. \end{aligned}$$

The limit terms vanish because $f' \in \text{HypI}$. We now use (7) and another integration by parts to yield

$$\begin{aligned}
k_{\mu}(B_{\mu}f) &= \frac{2p^{\frac{\mu+3}{2}}}{\Gamma(\mu+1)} \int_0^{\infty} \left(\frac{df}{dt}\right) t^{\frac{\mu+1}{2}} K_{\mu+1}(2\sqrt{pt}) dt \\
&= \frac{2p^{\frac{\mu+3}{2}}}{\Gamma(\mu+1)} \left\{ -\lim_{t \rightarrow 0^+} [f(t) t^{\frac{\mu+1}{2}} K_{\mu+1}(2\sqrt{pt})] + p^{\frac{1}{2}} \int_0^{\infty} f t^{\frac{\mu}{2}} K_{\mu}(2\sqrt{pt}) dt \right\} \\
&= pk_{\mu}(f)(p) - pf(0^+). \quad \text{QED}
\end{aligned}$$

This result immediately generalizes to the next theorem by induction.

THEOREM 3. If $f \in C^{2k}[0, \infty)$ and $f^{(2k-1)} \in \text{HypI}$, then

$$k_{\mu}(B_{\mu}^k f) = p^k k_{\mu}[f] - \sum_{j=1}^k p^j B_{\mu}^{k-j} f(0).$$

Note that this theorem is the integral transform version of Lemma 1 of [8]. The operational calculus for the operator B_{μ} is now effected through this formula. To solve the initial value problem

$$Q(B_{\mu})f(t) = g(t)$$

$$f(0) = C_0, B_{\mu}f(0) = C, \dots, B_{\mu}^{k-1}f(0) = C_{k-1} \quad (9)$$

where $Q(z)$ is a polynomial, we transform (9) into

$$Q(p)k_{\mu}f = P(p) + k_{\mu}g$$

where $P(p)$ is a polynomial of degree less than or equal to that of $Q(p)$. Therefore

$$k_{\mu}f = \frac{P(p)}{Q(p)} + \frac{1}{Q(p)} (k_{\mu}g)(p)$$

and $f(t)$ is retrieved by means of an inversion formula and

possibly a convolution theorem.

The following inversion theorem is obtained from Meijer's Theorem [18] through a simple change of variables, viz. $x \rightarrow \sqrt{t}$ and $y \rightarrow 2\sqrt{p}$.

THEOREM 4. Let μ be a complex number whose real part is not less than $-\frac{1}{2}$. Assume that in $\operatorname{Re}\sqrt{p} > \gamma_0 \geq 0$, $F(p)$ is an analytic function and is bounded according to $|F(p)| < M|p|^{-q}$ where $q > \frac{3}{2}\operatorname{Re}\mu + 2$. Then for real $c > \gamma_0$ and for $\operatorname{Re}\sqrt{p} > c$, $F(p) = k_\mu(f)$ where

$$f(t) = \frac{\Gamma(\mu+1)t^{-\frac{\mu}{2}}}{2\pi i} \int_{\operatorname{Re}\sqrt{p}=c} F(p)p^{-\frac{\mu}{2}-1} I_\mu(2\sqrt{pt}) dp. \quad (10)$$

The following lemma will be used in proving a convolution theorem for k_μ .

LEMMA. Letting D_∞^μ denote the Weyl derivative of order μ , we have

$$D_\infty^\mu \left\{ (z/t)^{\frac{\mu}{2}} K_\mu(2\sqrt{zt}) \right\} = (-z/t)^\mu K_{2\mu}(2\sqrt{zt}).$$

PROOF. By definition, $D_\infty^\mu \{f(t)\} = (d/dt)^k W^{k-\mu} \{f(t)\}$, where $k-1 < \mu < k$, and where

$$W^\nu \{f(t)\} = [-1/\Gamma(\nu)] \int_t^\infty f(s)(t-s)^{\nu-1} ds.$$

Since

$$K_\mu(2\sqrt{zt}) = \frac{1}{2}(t/z)^{\mu/2} \int_0^\infty \exp\{-(zy+t/y)\} y^{-\mu-1} dy \quad (11)$$

we have

$$\begin{aligned}
W^{k-\mu}\{t^{-\mu/2}K_{\mu}(2\sqrt{zt})\} &= \\
&= \{-z^{-\mu/2}/2\Gamma(k-\mu)\} \int_t^{\infty} (t-s)^{k-\mu-1} ds \int_0^{\infty} \exp\{-(zy+s/y)\} y^{-\mu-1} dy \\
&= \{-z^{-\mu/2}/2\Gamma(k-\mu)\} \int_0^{\infty} \exp(-zy) y^{-\mu-1} dy \int_t^{\infty} \exp(-s/y) \\
&\quad \cdot (t-s)^{k-\mu-1} ds.
\end{aligned}$$

On putting $s-t = y\lambda$, $ds = yd\lambda$ and using the definition of the gamma function, this becomes

$$(-1)^{k-\mu} \{z^{-\mu/2}/2\} \int_0^{\infty} \exp\{-(zy+t/y)\} y^{k-2\mu-1} dy.$$

Differentiating k times with respect to t , we get

$$\begin{aligned}
D_{\infty}^{\mu}\{t^{-\mu/2}K_{\mu}(2\sqrt{zt})\} &= \\
&= (-1)^{-\mu} \{z^{-\mu/2}/2\} \int_0^{\infty} \exp\{-(zy+t/y)\} y^{-2\mu-1} dy,
\end{aligned}$$

and using (11), with μ replaced by 2μ , completes the proof.

THEOREM 5. (Convolution theorem) If f and g belong to $C^{\infty}[0, \infty)$ and $f^{(n)}$ and $g^{(n)}$ satisfy HypI for every n , then $k_{\mu}(f \star g)$ converges absolutely in $\operatorname{Re} \sqrt{p} > \gamma_f + \gamma_g$ and

$$k_{\mu}(f \star g) = (k_{\mu}f)(k_{\mu}g) \text{ for } \mu \in (-1, \infty).$$

PROOF. We use the two-dimensional Laplace convolution theorem (pp. 26-29, [5]): Let

$$F_1(p, q) \equiv L(f_1) = \int_0^{\infty} \int_0^{\infty} e^{-px-qy} f_1(x, y) dx dy, \quad i = 1, 2$$

and

$$f^*(x, y) = \int_0^x \int_0^y f_1(\xi, \eta) f_2[x-\xi, y-\eta] d\eta d\xi. \quad (12)$$

If F_1 converges absolutely, then so does $F^*(p, q) \equiv L(f^*)$ and

$$F^*(p, q) = F_1(p, q) F_2(p, q). \quad (13)$$

In (12), let $f_1(x, y) = x^\mu f(xy)$ and $f_2(x, y) = x^\mu g(xy)$.

Then $f^*(x, y) = \frac{1}{y^{2\mu}} \zeta(xy)$ where

$$\zeta(t) = \int_0^t \int_0^1 w^\mu (t-w)^\mu f(wv) g[(t-w)(1-v)] dv dw, \text{ and}$$

$$\begin{aligned} L[f^*] &= \int_0^\infty \int_0^\infty e^{-px-xy} \zeta(xy) y^{-2\mu} dx dy \\ &= \int_0^\infty \zeta(t) \int_0^\infty e^{-qy-pty^{-1}} y^{-2\mu-1} dy dt. \end{aligned} \quad (14)$$

Similarly,

$$\begin{aligned} L[x^\mu f(xy)] &= \int_0^\infty f(t) dt \int_0^\infty \exp(-py-qty^{-1}) y^{\mu-1} dy \\ &= 2 \int_0^\infty f(t) p^{-\mu} (pqt)^{\frac{\mu}{2}} K_\mu(2\sqrt{pqt}) dt, \end{aligned} \quad (15)$$

on using the integral representation (5.2). Set $p = 1$ in (14) and (15) and invoke (13); we have

$$\int_0^\infty \zeta(t) dt \int_0^\infty \exp(-qy-ty^{-1}) y^{-2\mu-1} dy = \frac{\Gamma^2(\frac{\mu+1}{2})}{q^2} (k_\mu f)(k_\mu g). \quad (16)$$

It remains to show that the left hand side of (16) is equal to $\frac{\Gamma^2(\mu+1)}{q^2} k_\mu(f * g)$. Indeed, letting R_λ denote the Riemann-Liouville integral of order λ ,

Liouville integral of order λ ,

$$k_{\mu}(f * g) = k_{\mu} \left[\frac{1}{\Gamma(\mu+1)} B_{\mu} t^{-\mu} D^{\mu} \zeta(t) \right] \quad (17)$$

$$= \frac{q}{\Gamma(\mu+1)} k_{\mu} [t^{-\mu} D^{\mu} \zeta(t)] \quad (18)$$

$$= \frac{2q^2}{\Gamma^2(\mu+1)} \int_0^{\infty} (qt)^{\mu/2} K_{\mu}(2\sqrt{qt}) t^{-\mu} \left(\frac{d}{dt}\right)^k R_{k-\mu} \zeta(t) dt \quad (19)$$

$$= \frac{(-1)^k 2q^2}{\Gamma(k-\mu) \Gamma^2(\mu+1)} \int_0^{\infty} \left(\frac{d}{dt}\right)^k \{(q/t)^{\mu/2} K_{\mu}(2\sqrt{qt})\} dt \int_0^t \zeta(s) \cdot (t-s)^{k-\mu-1} ds \quad (20)$$

$$= \frac{(-1)^{\mu} 2q^2}{\Gamma^2(\mu+1)} \int_0^{\infty} \zeta(t) D_{\infty}^{\mu} \{(q/t)^{\mu/2} K_{\mu}(2\sqrt{qt})\} dt \quad (21)$$

$$= \frac{2q^2}{\Gamma^2(\mu+1)} \int_0^{\infty} \zeta(t) (q/t)^{\mu} K_{2\mu}(2\sqrt{qt}) dt \quad (22)$$

$$= \frac{q^2}{\Gamma^2(\mu+1)} \int_0^{\infty} \zeta(t) dt \int_0^{\infty} \exp(-qt - t/y) y^{-2\mu-1} dy \quad (23)$$

which proves our assertion concerning the left hand side of (16). Equation (17) follows from the definition of convolution. Equation (18) follows from theorem 2 since $Dt^{-\mu} D^{\mu} \zeta(t) \in \text{HypI}$ for $f^{(n)}$ and $g^{(n)} \in \text{HypI}$. Furthermore, from a theorem of Ritt [16], we have $f, g = O(1) \Rightarrow \zeta(t) = O(t^{2\mu+1}) \Rightarrow t^{-\mu} D^{\mu} \zeta(t) = O(t)$ as $t \rightarrow 0^+$. Thus $\lim_{t \rightarrow 0^+} t^{-\mu} D^{\mu} \zeta(t) = 0$. Equation (20) follows from (19) by the definition of the Riemann-Liouville integral, $R_{k-\mu}$, and integrating by parts k -times. The integrated terms vanish at $t = 0$ and $t = \infty$ by (6.1), (6.2), and the fact that in the definition of $\zeta(t)$, the functions f and g satisfy the hypotheses of theorem 5. Equation (22) is due to the preceding lemma. That

$k_{\mu}(f \star g)$ converges absolutely follows from the absolute convergence of (13). This completes the proof.

3. SOME OPERATIONAL FORMULAS.

$$\text{Let } F(y) = \int_0^{\infty} f(x) K_{\mu}(xy)^{1/2} dx \equiv \hat{f} \quad (24)$$

If we set $y = 2\sqrt{p}$ and $x = \sqrt{t}$, we get

$$k_{\mu}[f(\sqrt{t}) t^{-1/4-\mu/2}] = \frac{2^{2/3} p^{3/4+\mu/2}}{\Gamma(\mu+1)} F(2\sqrt{p}) . \quad (25)$$

From Erdélyi [7] p. 137 (16),

$$[x^{\beta+\mu-1/2} J_{\beta}(ax)]^{\wedge} = 2^{\beta+\mu} a^{\beta} y^{\mu+1/2} \Gamma(\beta+\mu+1) (y^2+a^2)^{-\beta-\mu-1} \quad (26)$$

$$\operatorname{Re} \beta > |\operatorname{Re} \mu| - 1, \operatorname{Re} y > |\operatorname{Im} p| .$$

By (25),

$$k_{\mu}[t^{\beta/2} J_{\beta}(2\sqrt{at})] = \frac{a^{\beta/2} p^{\mu+1}}{(p+a)^{\beta+\mu+1}} \cdot \frac{\Gamma(\beta+\mu+1)}{\Gamma(\mu+1)} , \quad (27)$$

$$\operatorname{Re} \sqrt{p} > |\operatorname{Im} \sqrt{a}| .$$

Letting $\beta = v - \mu$, this becomes, for $\operatorname{Re} v > \operatorname{Re} \mu$,

$$k_{\mu}[t^{(v-\mu)/2} J_{v-\mu}(2\sqrt{at})] = \frac{a^{(v-\mu)/2} p^{\mu+1} \Gamma(v+1)}{(p+a)^{v+1} \Gamma(\mu+1)} . \quad (28)$$

Since $K_{\mu}(\cdot) = K_{-\mu}(\cdot)$, we have from (26), for $\operatorname{Re} \sqrt{p} > |\operatorname{Im} a|$,

$$[x^{\beta-\mu+1/2} J_{\beta}(ax)]^{\wedge} = \frac{2^{\beta-\mu} a^{\beta} y^{-\mu+1/2} \Gamma(\beta-\mu+1)}{(y^2+a^2)^{-\mu+\beta+1}} . \quad (29)$$

From (25),

$$k_{\mu}[t^{\beta/2-\mu} J_{\beta}(2\sqrt{at})] = \frac{a^{\beta/2} \Gamma(\beta-\mu+1) p}{\Gamma(\mu+1) (p+a)^{\beta-\mu+1}} \quad (30)$$

Setting $\beta = v + \mu$,

$$k_{\mu} [t^{(v-\mu)/2} J_{v+\mu}(2\sqrt{at})] = \frac{\Gamma(v+1) a^{(v+\mu)/2} p}{\Gamma(\mu+1) (p+a)^{v+1}}. \quad (31)$$

If $v = 0$ in (31),

$$k_{\mu} [\Gamma(\mu+1) (at)^{-\mu/2} J_{\mu}(2\sqrt{at})] = \frac{p}{p+a}. \quad (32)$$

Letting $a \rightarrow -a$, and using $I_{\mu}(z) = e^{-\mu\pi i/2} J_{\mu}(iz)$,

$$k_{\mu} [\Gamma(\mu+1) (at)^{-\mu/2} I_{\mu}(2\sqrt{at})] = \frac{p}{p-a}. \quad (33)$$

Equation (31) can be written as

$$k_{\mu} \left[\frac{\Gamma(\mu+1)}{\Gamma(v+1)} t^v (at)^{-(v+\mu)/2} J_{v+\mu}(2\sqrt{at}) \right] = \frac{p}{(p+a)^{v+1}}. \quad (34)$$

Again letting $a \rightarrow -a$, and $I_{\mu}(z) = e^{-\mu\pi i/2} J_{\mu}(iz)$, this gives

$$k_{\mu} \left[\frac{\Gamma(\mu+1)}{\Gamma(v+1)} t^v (at)^{-(v+\mu)/2} I_{v+\mu}(2\sqrt{at}) \right] = \frac{p}{(p-a)^{v+1}}. \quad (35)$$

These expressions are useful in inverting rational functions.

As an application, consider the problem of solving

$$(B_{\mu}^2 + 3B_{\mu} + 2)\phi(t) = f(t),$$

$$\phi(0) = \phi_0,$$

$$(B_{\mu}\phi)(0) = \phi_1.$$

One gets

$$(p^2 + 3p + 2)k_{\mu}(\phi) = \phi_0 p^2 + (3\phi_0 + \phi_1)p + k_{\mu}f,$$

whence

$$k_{\mu}(\phi) = -\frac{(\phi_0 + \phi_1)p}{p+2} + \frac{(2\phi_0 + \phi_1)p}{p+1} + \left(\frac{1}{2} + \frac{p}{2(p+2)} - \frac{p}{p+1}\right)k_{\mu}f.$$

Therefore

$$\begin{aligned}\phi = & -(\phi_0 + \phi_1) \Gamma(\mu+1) (2t)^{-\mu/2} J_\mu(2\sqrt{2t}) \\ & + (2\phi_0 + \phi_1) \Gamma(\mu+1) t^{-\mu/2} J_\mu(2\sqrt{t}) + \frac{1}{2} f(t) \\ & + \left\{ \frac{1}{2} \Gamma(\mu+1) (2t)^{-\mu/2} J_\mu(2\sqrt{2t}) - \Gamma(\mu+1) t^{-\mu/2} J_\mu(2\sqrt{t}) \right\} * f(t).\end{aligned}$$

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