A NOTE ON RIEMANN INTEGRABILITY

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ABSTRACT. In this note we define Riemann integrability for real valued functions defined on a compact metric space accompanied by a finite Borel measure. If the measure of each open ball equals the measure of its corresponding closed ball, then a bounded function is Riemann integrable if and only if its set of points of discontinuity has measure zero.

Let $\mathcal{A}$ denote the algebra of sets generated by the open and closed subintervals of an interval $[a, b]$. A bounded real valued function f defined on $[a, b]$ is Riemann integrable if for each positive ϵ , there exist two functions ϕ and ψ that are linear combinations of characteristic functions of sets in $\mathcal{A}$ satisfying $\phi \leq f \leq \psi$ and

$$\int_a^b \psi \, dm - \int_a^b \phi \, dm < \epsilon$$

where m denotes ordinary Lebesgue measure. Riemann integrability may be defined in an analogous way for real valued functions defined on a compact metric space K accompanied by a finite Borel measure. If we make a simple

assumption about the balls of K , then the following famous theorem of Lebesgue extends: a bounded real valued function f defined on $[a,b]$ is Riemann integrable if and only if the set of points at which f is not continuous has Lebesgue measure zero.

Suppose that K is a compact metric space and μ is a finite Borel measure on K . Let $B_r(x) = \{y: d(x,y) < r\}$ and $\overline{B}_r(x) = \{y: d(x,y) \leq r\}$ denote the open and closed balls of radius r about a point x in K . Let $\mathcal{A}$ denote the algebra generated by all such balls. Any element of $\mathcal{A}$ is of the form

$$\bigcup_{1 \leq i \leq m} \bigcap_{1 \leq k \leq n_i} A_{ik} \quad (1)$$

where A_{ik} is a ball or its complement and $\{m, n_1, \dots, n_m\}$ are positive integers. A step function is a linear combination of characteristic functions determined by elements of $\mathcal{A}$. Hence a step function ϕ has the form $\sum d_i \chi_{A_i}$ where each d_i is real and $A_i \in \mathcal{A}$. Since $\mathcal{A}$ is an algebra, the $\{A_i\}$ can be taken to be pairwise disjoint. It is easy to see that if ϕ and ψ are step functions, then so are $\phi + \psi$, $\phi - \psi$, $\inf \{\phi, \psi\}$, and $\sup \{\phi, \psi\}$.

DEFINITION. A bounded real valued function f defined on K is Riemann integrable if for each positive ϵ there exist step functions ϕ and ψ such that $\phi \leq f \leq \psi$ and $\int \psi d\mu - \int \phi d\mu < \epsilon$.

Given a bounded real valued function f defined on K , the upper envelope h of f is the function defined by

$$h(x) = \inf_{\delta > 0} \sup_{y \in B_\delta(x)} f(y) \quad x \in K$$

and the lower envelope g of f is defined by

$$g(x) = \sup_{\delta > 0} \inf_{y \in B_\delta(x)} f(y) \quad x \in K$$

It is well known that h is upper semicontinuous, g is lower semicontinuous, $g(x) \leq f(x) \leq h(x)$ for each x , and $g(x) = h(x)$ if and only if f is continuous at x (see Royden [1, p.49]).

THEOREM. Suppose $\mu(B_r(x)) = \mu(\overline{B_r}(x))$ for each x in K and for each positive r . A bounded real valued function f defined on K is Riemann integrable if and only if the set of points at which f is discontinuous has μ -measure zero.

Proof. Let h be the upper envelope of f and g its lower envelope. Let ψ be any step function that exceeds f . Since each member of $\mathcal{A}$ can be expressed in the form depicted in (1), the condition on the balls of K implies that each member of $\mathcal{A}$ is the union of an open set and a set of μ -measure zero. It follows that ψ can be represented as

$$\sum_{j=1}^n a_j \chi_{A_j}$$

where (i) A_j is an open set for $1 \leq j \leq m$ (ii) $\mu(A_j) = 0$ for $m < j \leq n$ (iii) $\{A_1, A_2, \dots, A_n\}$ partition K .

Let $x \in \bigcup_{j=1}^m A_j$. Since ψ is constant near x , there exists $\delta > 0$ such that $\psi(x) \geq \sup_{y \in B_\delta(x)} f(y)$ so that $\psi(x) \geq h(x)$. Hence, $\mu\{x: \psi(x) < h(x)\} = 0$, and we have $\int \psi \, d\mu \geq \int h \, d\mu$. We now construct a decreasing sequence of step functions converging pointwise to h so that

$\inf \{ \int \psi \, d\mu : \psi \geq f \text{ and } \psi \text{ is a step function} \} = \int h \, d\mu.$

Let N be a fixed positive integer. Let $\{B_{r_1}(x_1), \dots, B_{r_m}(x_m)\}$ be a cover of K by balls of radius at most $1/N$ such that if $y \in B_{r_i}(x_i)$, then $h(y) < h(x_i) + 1/N$. Now let $\theta_N: K \rightarrow \mathbb{R}$ be the step function described by $\theta_N(x) = \inf \{h(x_i) + 1/N : x \in B_{r_i}(x_i)\}$. Define ψ_N to be θ_N . Given any positive integer p , define θ_{N+p} as above, and let ψ_{N+p} be $\inf \{\theta_{N+p}, \psi_{N+p-1}\}$. Clearly, for each p ψ_{N+p} is a step function, and $\psi_{N+p} \geq \psi_{N+p+1} \geq h$. To establish the pointwise convergence, suppose to the contrary that for some x_0 in K and $\epsilon > 0$ we have for each p

$$\psi_{N+p}(x_0) > h(x_0) + 2\epsilon$$

Pick n so large that $1/n < \epsilon$. There exists a point x_n such that $d(x_0, x_n) < 1/n$ and $\psi_n(x_0) \leq h(x_n) + 1/n$. Clearly, $h(x_n) > h(x_0) + \epsilon$ which violates the upper semicontinuity of h . Hence, $\{\psi_n\}$ is the desired sequence.

Using the above technique we can show in the same manner that $\int g \, d\mu = \sup \{ \int \phi \, d\mu : \phi \leq f \text{ and } \phi \text{ is a step function} \}$. The proof is now completed by observing the equivalence of the following statements: (i) f is Riemann integrable (ii) $\int g \, d\mu = \int h \, d\mu$ (iii) f is continuous except at a set of points of μ -measure zero.

A simple example shows that the theorem need not hold if our condition on the balls of the metric space is omitted. Let K be the closed unit disc in the plane with the usual metric. If B is a Borel subset of K , define $\mu(B)$ to be $\mu_1(B \cap \{(x, y) : x^2 + y^2 = 1\}) + \mu_2(B \cap \{(x, y) : x^2 + y^2$

$< 1\}$ where μ_2 is two dimensional Lebesgue measure and μ_1 is one dimensional Lebesgue measure, considering the circle as having measure 2π . Then the characteristic function of the unit circle is Riemann integrable (being a step function), but its set of discontinuities has measure 2π .

REFERENCES

1. H. L. Royden. Real Analysis, Macmillan, New York, 1968.

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