RESEARCH NOTES

ON VON NEUMANN'S INEQUALITY

ARTHUR LUBIN

Department of Mathematics
Illinois Institute of Technology
Chicago, Illinois 60616

(Received October 31, 1977)

Von Neumann's inequality states that for a contraction T acting on a Hilbert space H

$$(v) \quad ||p(T)|| \leq \sup \{|p(z)| : |z| < 1\}$$

holds for all polynomials p . The analog for a set of commuting contractions $\{T_1, \dots, T_n\}$,

$$(v_n) \quad ||p(T_1, \dots, T_n)|| \leq \sup \{|p(z_1, \dots, z_n)| : |z_1| < 1\}$$

is known to be false for $n > 2$. In fact, for any $c > 0$, there exist $\{T_1, \dots, T_n\}$, where n is sufficiently large, and a polynomial p such that

$$||p(T_1, \dots, T_n)|| > c \sup \{|p(z_1, \dots, z_n)| : |z_1| < 1\}, \quad (2)$$

In this note we establish the following weakened version of (v_n) :

PROPOSITION 1. Let $\{T_1, \dots, T_n\}$ be commuting contractions on a Hilbert space H .

Then for any polynomial p ,

$$||p(T_1, \dots, T_n)|| \leq \sup \{ |p(z_1, \dots, z_n)| : |z_i| < n^{\frac{1}{2}} \},$$

i.e., $D_n = \{(z_1, \dots, z_n) : |z_i| < n^{\frac{1}{2}}\}$ is a spectral set for $(T_1, \dots, T_n)$.

Our proof is an easy consequence of the following proposition.

PROPOSITION 2 (3, I.9.2). Let $\{S_1, \dots, S_n\}$ be commuting contractions with $\sum_{i=1}^n ||S_i||^2 \leq 1$. Then $\{S_1, \dots, S_n\}$ has a commuting unitary dilation (in fact a regular one) and it therefore follows immediately that $\{S_1, \dots, S_n\}$ satisfies (v_n) .

PROOF OF PROPOSITION 1. Given $\{T_1, \dots, T_n\}$, let $S_i = n^{-\frac{1}{2}} T_i$, $i = 1, \dots, n$.

Then $\sum_{i=1}^n ||S_i||^2 = n^{-1} \sum_{i=1}^n ||T_i||^2 \leq 1$ so (v_n) holds for $\{S_1, \dots, S_n\}$.

Given any polynomial $p(z_1, \dots, z_n)$, let $q(z_1, \dots, z_n) = p(n^{\frac{1}{2}} z_1, \dots, n^{\frac{1}{2}} z_n)$.

Then

$$\begin{aligned} ||p(T_1, \dots, T_n)|| &= ||p(n^{\frac{1}{2}} S_1, \dots, n^{\frac{1}{2}} S_n)|| \\ &= ||q(S_1, \dots, S_n)|| \\ &\leq \sup \{ |q(w_1, \dots, w_n)| : |w_i| < 1 \} \\ &= \sup \{ |p(n^{\frac{1}{2}} w_1, \dots, n^{\frac{1}{2}} w_n)| : |w_i| < 1 \} \\ &= \sup \{ |p(z_1, \dots, z_n)| : |z_i| < n^{\frac{1}{2}} \} \end{aligned}$$

COROLLARY 3. (see (1) p. 279). Any set $\{T_1, \dots, T_n\}$ of commuting contractions on H has the polydisc $D_n = \{(z_1, \dots, z_n) : |z_i| < n^{\frac{1}{2}}\}$ as a complete spectral set.

PROOF. By proposition 2, there exist commuting unitary operators $U_1, \dots, U_n$ on a Hilbert space K containing H such that $q(S_1, \dots, S_n) = P q(U_1, \dots, U_n)$ for all polynomials q , where $S_i = n^{-\frac{1}{2}} T_i$ and P projects K onto H . Setting $N_i = n^{\frac{1}{2}} U_i$, we have that $\{N_1, \dots, N_n\}$ is a normal dilation of $\{T_1, \dots, T_n\}$ with joint spectrum $\text{sp}(N)$ contained in the boundary of D_n and the corollary follows as in (1).

Similarly, it follows that $D_a = \{(z_1, \dots, z_n) : |z_i| < a_i\}$ is a complete spectral set for all commuting contractions $\{T_1, \dots, T_n\}$ if $\sum a_i^{-2} < 1$.

Since the common intersection of such D_a is the unit polydisc D , which is not in general a complete spectral set since (v_n) can fail if $n \geq 3$, we have

COROLLARY 4. If $\{T_1, \dots, T_n\}$ is a set of commuting contractions such that the intersection of any two complete spectral sets is also a complete spectral set, then the unit polydisc D is also a complete spectral set.

We note that von Neumann's original paper (4) showed that for a single contraction the intersection of two spectral sets need not be a spectral set.

Since (v_n) holds for $n = 2$, we see that proposition 1 is not the best possible result. This prompts the following

PROBLEM. Find

$$V(n) = \inf\{r: \|p(T_1, \dots, T_n)\| \leq \sup\{|p(z_1, \dots, z_n)|: |z_i| < r\}\}$$

We note that Theorem 1.2(b) of (2) yields information concerning the growth of $V(n)$ as n increases. Since

$$\sup\{|p(z_1, \dots, z_n)|: |z_i| < r\} = r^s \sup\{|p(z)|: |z_i| < 1\}$$

for homogeneous polynomials of degree s , we have for any $\varepsilon > 0$, $V(n) \geq n^{(1/4-\varepsilon)}$ for n sufficiently large.

ACKNOWLEDGMENT. This research was partially supported by NSF Grant MCS 76-06516 A01.

REFERENCES

1. Arveson, W. Subalgebras of C^* -algebras II, Acta. Math. 128 (1972) 271-308.
2. Dixon, P. G. The von Neumann Inequality for Polynomials of Degree Greater Than 2, J. London Math. Soc. 14 (1976) 369-375.
3. Sz.-Nagy, B. and C. Foias. Harmonic Analysis of Operators on Hilbert Space, North-Holland, 1970.
4. von Neumann, J. Eine Spektraltheorie für Allgemeine Operatoren Eines Unitären Raumes, Math. Nachr. 4 (1951) 258-281.

KEY WORDS AND PHRASES. *Commuting contraction on a Hilbert Space, spectral set, Inequality.*

AMS(MOS) SUBJECT CLASSIFICATIONS(1970). 47A25.