ON SEPARABLE EXTENSIONS OF GROUP RINGS AND QUATERNION RINGS

GEORGE SZETO

Mathematics Department
Bradley University
Peoria, Illinois 61625 U.S.A.

(Received November 29, 1977)

ABSTRACT. The purposes of the present paper are (1) to give a necessary and sufficient condition for the uniqueness of the separable idempotent for a separable group ring extension RG (R may be a non-commutative ring), and (2) to give a full description of the set of separable idempotents for a quaternion ring extension RQ over a ring R , where Q are the usual quaternions i, j, k and multiplication and addition are defined as quaternion algebras over a field. We shall show that RG has a unique separable idempotent if and only if G is abelian, that there are more than one separable idempotents for a separable quaternion ring RQ , and that RQ is separable if and only if 2 is invertible in R .

KEY WORDS AND PHRASES. Group Rings, Idempotents in Rings, Separable Algebras

AMS (MOS) SUBJECT CLASSIFICATION (1970) CODES. Major: 16A26 Minor: 16A16,
16A32

1. INTRODUCTION.

M. Auslander and O. Goldman ([1] and [2]) studied separable algebras over a commutative ring. Subsequently, the investigation of separable algebras (in particular, Brauer groups and Azumaya algebras) has attracted a lot of researchers, and rich results have been obtained (see References). K. Hirata and K. Sugano ([5]) generalized the concept of separable algebras to separable ring extensions; that is, let S be a subring of a ring T with the same identity. Then T is called a separable ring extension of S if there exists an element $\sum a_i \otimes b_i$ in $T \otimes_S T$ such that $x(\sum a_i \otimes b_i) = (\sum a_i \otimes b_i)x$ for each x in T and $\sum a_i b_i = 1$. Such an element $\sum a_i \otimes b_i$ is called a separable idempotent for T . We note that a separable idempotent takes an important role in many theorems (for example, see [6], Section 5, 6, and 7). It is easy to verify that $(1/n)(\sum g_i \otimes g_i^{-1})$ and $\sum e_{i1} \otimes e_{1i}$ ([4], Examples II and III, P. 41) are separable idempotents for a group algebra RG and a matrix ring $M_m(R)$ respectively, where $G = \{g_1, \dots, g_n\}$ with n invertible in R and e_{ij} are matrix units. We also note that the separable idempotent for a commutative separable algebra is unique ([6], Section 1, P. 722).

2. PRELIMINARIES.

Throughout, G is a group of order n , R is a ring with an identity

1. The group ring $RG = \{\sum r_i g_i \mid r_i \text{ in } R \text{ and } g_i \text{ in } G\}$, which is a free R -module with a basis $\{g_i\}$ and $(\sum r_i g_i)(\sum s_i g_i) = \sum t_k g_k$ where $t_k = \sum r_i s_j$ for all possible i, j such that $g_i g_j = g_k$. The ring R is imbedded in RG by $r \mapsto r g_1$, where g_1 is the identity of G ($g_1 = 1$). The multiplication map $RG \otimes_R RG \rightarrow RG$ is denoted by π . Clearly, $\{g_i \otimes g_j \mid i, j =$

$1, \dots, n\}$ form a basis for $RG \otimes_R RG$. An element $\sum r_{ij}(\varepsilon_i \otimes \varepsilon_j)$ in $RG \otimes_R RG$ is called a commutant element in $RG \otimes_R RG$ if $x(\sum r_{ij}(\varepsilon_i \otimes \varepsilon_j)) = (\sum r_{ij}(\varepsilon_i \otimes \varepsilon_j))x$ for all x in RG .

3. MAIN THEOREMS.

We begin with a representation for $\pi(x)$ for a commutant element x in $RG \otimes_R RG$, and then we show that RG has a unique separable idempotent if and only if G is abelian.

LEMMA 1. Let $x = \sum r_{ij}(\varepsilon_i \otimes \varepsilon_j)$, $i, j = 1, \dots, n$, be a commutant element in $RG \otimes_R RG$. Then $\pi(x) = \sum_{i=1}^m (\sum r_{1k'_i})_{n_{k'_i}} C_{k'_i}$, where m is the number of conjugate classes of G , $n_{k'_i}$ is the order of the normalizer of $\varepsilon_{k'_i}$, and $C_{k'_i}$ is the sum of different conjugate elements of $\varepsilon_{k'_i}$, for some k'_i and k'_i in $\{1, \dots, n\}$.

PROOF. Since x is a commutant element, $\varepsilon_p x = x \varepsilon_p$ for each ε_p in G . The coefficient of the term $\varepsilon_p \otimes \varepsilon_k$ in $\varepsilon_p x$ is r_{1k} , and the coefficient of the same term in $x \varepsilon_p$ is r_{pq} , where $\varepsilon_q \varepsilon_p = \varepsilon_k$. Hence $r_{1k} = r_{pq}$ whenever $\varepsilon_q \varepsilon_p = \varepsilon_k$. Thus $x = \sum_k r_{1k} (\sum \varepsilon_p \otimes \varepsilon_q)$, where p, q run over $1, \dots, n$, such that $\varepsilon_q \varepsilon_p = \varepsilon_k$; that is, $x = \sum_k r_{1k} (\sum \varepsilon_p \otimes \varepsilon_k \varepsilon_p^{-1})$. Taking $\pi(x) = \sum_k r_{1k} (\sum \varepsilon_p \varepsilon_k \varepsilon_p^{-1})$. For a fixed k , $\sum \varepsilon_p \varepsilon_k \varepsilon_p^{-1} = n_k C_k$ where n_k is the order of the normalizer of ε_k and C_k is the sum of all different conjugate elements of ε_k . Hence $\pi(x) = \sum_{k=1}^n r_{1k} n_k C_k$. Since conjugate classes form a partition of G , $C_i = C_j$ if and only if ε_i is conjugate to ε_j . Renumerating elements, we let $\{\varepsilon_{k_1}, \dots, \varepsilon_{k_m}\}$ be all non-conjugate elements of each other; then $\{C_{k_1}, \dots, C_{k_m}\}$ are all different elements in the set, $\{C_1, \dots, C_n\}$. Thus $\pi(x) = \sum_{i=1}^m (\sum r_{1k'_i})_{n_{k'_i}} C_{k'_i}$, where $r_{1k'_i}$ are coefficients of the same $C_{k'_i}$, and m is the number of conjugate classes

of G .

THEOREM 2. Let RG be a separable extension of R . Then, RG has a unique separable idempotent if and only if G is abelian.

PROOF. Let $x = \sum r_{ij}(g_i \otimes g_j)$ be a separable idempotent for RG . Then by the lemma, $\pi(x) = \sum_{i=1}^m (\sum r_{1k_i}) n_{k_i} C_{k_i}$, where C_{k_i} is the sum of all conjugate elements of g_{k_i} . Let $g_{k_1} = 1$, the identity of G . Then $C_{k_1} = 1$ and $n_{k_1} = n$, the order of G . Since $\pi(x) = 1$, $(\sum r_{1k_1}) n_{k_1} C_{k_1} = 1$ and $(\sum r_{1k_1}) n_{k_1} C_{k_1} = 1$ and $(\sum r_{1k_i}) n_{k_i} C_{k_i} = 0$ for each $i \neq 1$. Noting that $C_{k_1} = 1$, we have $\sum r_{1k_1} = r_{11}$, and so the first equation becomes $r_{11}n = 1$. Hence the order of G , n , is invertible in R . Thus n_{k_i} , being a factor of n , is also invertible in R . But conjugate classes form a partition of G , so $(\sum r_{1k_i}) n_{k_i} C_{k_i} = 0$ implies that $\sum r_{1k_i} = 0$ for each $i \neq 1$. This system of homogeneous equations $\sum r_{1k_i} = 0$ in the unknowns r_{1k_i} with $i \neq 1$ has trivial solutions if and only if $n = m$, and this holds if and only if G is abelian. Since the uniqueness of the separable idempotent $(= (1/n)(\sum g_i \otimes g_i^{-1}))$ is equivalent to the existence of trivial solutions of the above system of equations, the same fact is equivalent to G being abelian.

The theorem tells us that there are many separable idempotents for a separable group ring RG when G is non-abelian. Also, we remark that if RG is a separable extension of R , the order of G is invertible in R from the proof of the theorem. Next, we discuss another popular separable ring extension, a quaternion ring extension RQ , where $RQ = \{r_1 + r_2i + r_3j + r_4k \mid i, j, \text{ and } k \text{ are usual quaternions}\}$. $(RQ, +, \cdot)$ is a ring extension of R under the usual addition and multiplication similar to quaternion algebras over a field. Now we characterize a separable idem-

potent for a separable quaternion ring extension RQ .

THEOREM 3. Let RQ be a separable quaternion ring extension. Then a commutant element $x = \sum r_{st}(s\otimes t)$, $s, t = 1, i, j, k$, in $RQ \otimes_R RQ$ is a separable idempotent for RQ if and only if $r_{11} = 1/4$.

PROOF. Since x is a commutant element in $RQ \otimes_R RQ$, $ix = xi$. The coefficients of the term $1\otimes 1$ on both sides are $-r_{i1}$ and $-r_{1i}$, so $r_{i1} = r_{1i}$. Since $jx = xj$, the coefficients of the term $k\otimes 1$ on both sides are $-r_{i1} = -r_{kj}$, so $r_{i1} = r_{kj}$. Also, $kx = xk$, so the coefficients of the term $j\otimes 1$ on both sides are $-r_{i1} = r_{jk}$. Hence $r_{1i} = r_{i1} = r_{kj} = -r_{jk}$. Similarly, by comparing coefficients of other terms, we have $r_{11} = -r_{ii} = -r_{jj} = -r_{kk}$, $r_{1j} = r_{j1} = -r_{ki} = r_{ik}$ and $r_{1k} = r_{k1} = -r_{ij} = r_{ji}$. In other words, $r_{st} = r_{pq}$ if $ts = qp$, and $r_{st} = -r_{pq}$ if $ts = -qp$. Thus

$$x = r_{11}(1\otimes 1 - i\otimes i - j\otimes j - k\otimes k) + r_{1i}(1\otimes i + i\otimes 1 - j\otimes k + k\otimes j) + r_{1j}(1\otimes j + j\otimes 1 - k\otimes i + i\otimes k) + r_{1k}(1\otimes k + k\otimes 1 - i\otimes j + j\otimes i) \dots (*)$$

But then $\pi(x) = r_{11}^4 + r_{1i}^0 + r_{1j}^0 + r_{1k}^0 = 4r_{11}$. Consequently, x is a separable idempotent if and only if $r_{11} = 1/4$ (for $\pi(x) = 1$).

COROLLARY 4. Let RQ be a quaternion ring extension of R . Then RQ is separable if and only if 2 is invertible in R .

PROOF. The necessity is immediate from the theorem. The sufficiency is clear since the element x with $r_{11} = 1/4$, $r_{1i} = r_{1j} = r_{1k} = 0$ as given in (*) in Theorem 3 is a separable idempotent for RQ .

REMARK. It is easy to see that every x of the form (*) in Theorem 3 with r_{11} , r_{1i} , r_{1j} and r_{1k} in the center of R is a commutant element in $RQ \otimes_R RQ$. Hence, from the proof of Theorem 3, the complete set of commutant elements is: $C = \{ \sum r_{st}(s\otimes t) / r_{st} = r_{pq} \text{ if } qp = ts, \text{ and } r_{st} = -r_{pq} \text{ if } qp = -ts \}$. Also, the complete set of separable idempotents for

RQ is a subset of C such that $r_{11} = 1/4$ and r_{1i}, r_{1j}, r_{1k} are in the center of R . Thus there are many separable idempotents.

REFERENCES

1. Auslander, M. and O. Goldman. The Brauer Group of a Commutative Ring, Trans. Amer. Math. Soc. 97 (1960) 367-409.
2. Auslander, M. and O. Goldman. Maximal Orders, Trans. Amer. Math. Soc. 97 (1960) 1-24.
3. Bass, H. Lectures on Topics in Algebraic K-Theory, Tata Institute of Fundamental Research, Bombay, 1967.
4. DeMeyer, F. and E. Ingraham. Separable Algebras Over Commutative Rings, Springer-Verlag, Berlin-Heidelberg-New York, 181, 1971.
5. Hirata, K. and K. Sugano. On Semisimple Extensions and Separable Extensions over Non-Commutative Rings, J. Math. Soc. Japan 18 (1966) 360-373.
6. Villamayor, O. and D. Zelinsky. Galois Theory for Rings with Finitely Many Idempotents, Nagoya Math. J. 27 (1966) 721-731.
7. Zelinsky, D. Brauer Groups, Springer-Verlag, Berlin-Heidelberg-New York, 549, 1976.