MONOPOLE HARMONICS FOR A SPECIAL GAUGE

ZENAIDA E. S. UY

The Pennsylvania State University
Shenango Valley Campus
Sharon, Pennsylvania 16146

(Received April 18, 1978)

ABSTRACT. The monopole harmonics for a special gauge are derived and found to differ from the Wu-Yang monopole harmonics by a factor of $e^{iq\gamma \cos\theta}$, where γ is a real parameter.

KEYWORDS AND PHRASES. Monopole harmonics, Schrödinger Equation, angular momentum operators, Classical Gauge Field Theory, Monopole Theory.

AMS (MOS) SUBJECT CLASSIFICATION (1970) CODES. 81A39, 78A25.

1. INTRODUCTION.

With help of concepts incorporated from the mathematics of fibre bundles, Wu and Yang [1] have shown that the wave function ψ of a particle of charge Ze around a Dirac monopole of strength g should be regarded as a section which do not have discontinuities. The eigensections of the angular momentum operators, which they appropriately called monopole harmonics, were explicitly constructed.

The gauge [1,2] they have chosen was specified by:

(1) the regions

$$R_a : 0 \leq \theta < \frac{1}{2} \pi + \delta \quad r > 0 \quad 0 \leq \phi < 2\pi \quad (1.1)$$

$$R_b : \frac{1}{2} \pi - \delta < \theta \leq \pi \quad r > 0 \quad 0 \leq \phi < 2\pi \quad (1.2)$$

$$R_{ab} : \frac{1}{2} \pi - \delta < \theta < \frac{1}{2} \pi + \delta \quad r > 0 \quad 0 \leq \phi < 2\pi \quad (1.3)$$

where δ was chosen such that $0 \leq \delta \leq \frac{1}{2} \pi$,

(2) the vector potentials

$$(A_r)_a = (A_\theta)_a = 0 \quad (A_\phi)_a = \frac{g}{r \sin \theta} (1 - \cos \theta) \quad (1.4)$$

$$(A_r)_b = (A_\theta)_b = 0 \quad (A_\phi)_b = \frac{-g}{r \sin \theta} (1 + \cos \theta) \quad (1.4a)$$

where A_r , A_θ , A_ϕ are the spherical components of the vector potential $\underline{A}$; r , θ , ϕ are the spherical coordinates; and the subscripts a and b denote the regions R_a and R_b respectively,

(3) the gauge transformation factor S in the overlap region R_{ab} :

$$S = S_{ab} = e^{2iq\phi} \quad (1.5)$$

$$\psi_a = S_{ab} \psi_b \quad (1.6)$$

$$(A_\mu)_a = (A_\mu)_b + \frac{1}{Ze} S_{ab} \frac{\partial S_{ab}^{-1}}{\partial x^\mu} \quad (1.7)$$

Here $q = \frac{1}{2} DZ$, where $D = 2eg$ is the Dirac monopole strength in units where $\hbar = c = 1$; D and q are integers which may be positive, negative or zero, and ψ_a and ψ_b are the wave function of a particle of charge Ze in R_a and R_b respectively.

The Schrödinger equations for a particle of charge Ze in the field of a monopole with spherically symmetric potential $V(r)$ are:

$$\frac{1}{2m} (\underline{p} - Ze\underline{A}_a)^2 \psi_a + V\psi_a = E\psi_a \quad \text{in } R_a \quad (1.8)$$

$$\frac{1}{2m} (\underline{p} - Ze\underline{A}_b)^2 \psi_b = V\psi_b = E\psi_b \quad \text{in } R_b \quad (1.9)$$

Wu and Yang have shown that $\underline{r}$, $(\underline{p} - Ze\underline{A})$, and the L_1 , L_2 , L_3 components of

the operator

$$\tilde{L} = \tilde{r} \times (\tilde{p} - Ze\tilde{A}) - q \frac{\tilde{r}}{\tilde{r}} \quad (1.10)$$

are Hermitian operators in the Hilbert space of sections and that they satisfy the commutation relations

$$[L_i, X_j] = i \epsilon_{ijk} X_k \quad (1.11)$$

$$[L_i, p_j - ZeA_j] = i \epsilon_{ijk} (p_k - ZeA_k) \quad (1.12)$$

$$[L_i, L_j] = i \epsilon_{ijk} L_k \quad (1.13)$$

(where ϵ_{ijk} is the Levi-Civita symbol) which show that the $\tilde{L}$ in Eg. (1.10) is the angular momentum operator.

Because $[\tilde{r}^2, \tilde{L}] = 0$, it is possible to separate the angular part of a section from its radial part. The angular sections which are simultaneous eigensections of $\tilde{L}^2$ and L_z are called the monopole harmonics and are denoted by $Y_{q,\ell,m}(\theta, \phi)$. Explicitly, they are

$$\left[Y_{q,\ell,m} \right]_a = M_{q,\ell,m} (1-x)^{\frac{\alpha}{2}} (1+x)^{\frac{\beta}{2}} P_n^{\alpha,\beta}(x) e^{i(m+\phi)} \quad (1.14)$$

$$\left[Y_{q,\ell,m} \right]_b = \left[Y_{q,\ell,m} \right]_a e^{-2iq\phi} \quad (1.15)$$

where

$$x = \cos\theta, \quad \alpha = -q-m, \quad \beta = q-m, \quad n = \ell+m \quad (1.16)$$

$$M_{q,\ell,m} = 2^m \left[\frac{(2\ell+1)}{4\pi} \cdot \frac{(\ell-m)! (\ell+m)!}{(\ell-q)! (\ell+q)!} \right]^{\frac{1}{2}} \quad (1.17)$$

and $P_n^{\alpha,\beta}(x)$ are the Jacobi polynomials:³

$$P_n^{\alpha,\beta}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^n}{dx^n} \left[(1-x)^{\alpha+m} (1+x)^{\beta+m} \right] \quad (1.18)$$

These harmonics satisfy

$$L^2 Y_{q,\ell,m} = \ell(\ell+1) Y_{q,\ell,m} \quad (1.19)$$

$$L_3 Y_{q,\ell,m} = m Y_{q,\ell,m} \quad (1.19a)$$

For each q , the allowed values of ℓ and m are

$$\ell = |q|, |q| + 1, |q| + 2, \dots \quad (1.20)$$

$$m = -\ell, -\ell+1, \dots, +\ell \quad (1.20a)$$

Wu and Yang also proved that for a fixed q , the set $Y_{q,\ell,m}$ form a complete set of orthonormal sections and that they are analytic everywhere.

It is the aim of this paper to derive the monopole harmonics consistent with a different vector potential.

2. THE SPECIAL GAUGE WITH DIFFERENT VECTOR POTENTIAL,

Consider the vector potential $\hat{\tilde{A}}$ whose components are

$$\begin{aligned} (\hat{\tilde{A}}_\theta)_a &= \frac{-\gamma g \sin\theta}{r} & (\hat{\tilde{A}}_\phi)_a &= \frac{g}{r \sin\theta} (1-\cos\theta) & (\hat{\tilde{A}}_r)_a &= 0 \\ (\hat{\tilde{A}}_\theta)_b &= \frac{-\gamma g \sin\theta}{r} & (\hat{\tilde{A}}_\phi)_b &= \frac{-g}{r \sin\theta} (1+\cos\theta) & (\hat{\tilde{A}}_r)_b &= 0 \end{aligned} \quad (2.1a\&b)$$

where a and b denote the regions R_a and R_b respectively; γ is a parameter.

Seperately, $\hat{\tilde{A}}_a$ and $\hat{\tilde{A}}_b$ are still monopole fields in spite of the addition of a non-zero component $\hat{\tilde{A}}_\theta$:

$$\begin{aligned} \nabla \times \hat{\tilde{A}}_a &= \tilde{B}_a = \frac{g}{r^3} \tilde{r} \\ \nabla \times \hat{\tilde{A}}_b &= \tilde{B}_b = \frac{g}{r^3} \tilde{r} \end{aligned} \quad (2.2a\&b)$$

In the special gauge which is characterized by the same regions R_a , R_b , R_{ab} ; the same gauge transformation phase factor $S_{ab} = e^{2iq\phi}$; but with the above

modified vector potential (Eq. 2.1a&b), we find the following:

$$(i) \quad \underline{r}, \underline{p} - Ze\hat{A}, \underline{L} = \underline{r} \times (\underline{p} - Ze\hat{A}) - \frac{q}{r} \underline{r} \quad \text{remain Hermitian operators}$$

satisfying Eqs. (1.11), (1.12), and (1.13) provided γ is real. Hence

$$\underline{L} = \underline{r} \times (\underline{p} - Ze\hat{A}) - \frac{q}{r} \underline{r} \quad \text{remains the angular momentum poerator.}$$

(ii) The resulting monopole harmonics, which we shall denote by $\hat{Y}_{q,\ell,m}$, are the same as the monopole harmonics of Wu and Yang except for a factor of $e^{iq\gamma \cos\theta}$:

$$\begin{aligned} \hat{Y}_{q,\ell,m}{}_a &= e^{iq\gamma \cos\theta} \left[Y_{q,\ell,m} \right]_a \\ \hat{Y}_{q,\ell,m}{}_b &= e^{iq\gamma \cos\theta} \left[Y_{q,\ell,m} \right]_b \end{aligned}$$

(i) means ℓ and m satisfy Eq. (1.20a). (ii) implies that $Y_{q,\ell,m}$ form a complete, analytic orthogonal set for each q and that the possible values of ℓ are those in Eq. (1.20).

3. THE DERIVATION OF $\hat{Y}_{q,\ell,m}$

The steps used in deriving the explicit expression for $\hat{Y}_{q,\ell,m}$ will be described since the technique used in obtaining the solution of the differential equation was different from those of Wu and Yang.

From the expression of the angular momentum

$$\underline{L} = \underline{r} \times (\underline{p} - Ze\hat{A}) - \frac{q}{r} \underline{r} \quad (3.1)$$

it is straightforward to show that

$$m\hat{Y}_{q,\ell,m} = L_3\hat{Y}_{q,\ell,m} = (-\partial\phi - q) \hat{Y}_{q,\ell,m} \quad \text{in } R_a \quad (3.2)$$

$$m\hat{Y}_{q,\ell,m} = L_3\hat{Y}_{q,\ell,m} = (-i\partial\phi + q) Y_{q,\ell,m} \quad \text{in } R_b \quad (3.2a)$$

so that

$$\left[\hat{Y}_{q,\ell,m} \right]_a = \left[\hat{\Theta}_{q,\ell,m}(\theta) \right]_a e^{i(m+q)\phi} \quad (3.3)$$

$$\left[\hat{Y}_{q,\ell,m} \right]_b = \left[\hat{\Theta}_{q,\ell,m}(\theta) \right]_b e^{i(m+q)\phi} \quad (3.3a)$$

From Eqs. (1.5) and (1.6)

$$\left[\hat{\Theta}_{q,\ell,m} \right]_a = \left[\hat{\Theta}_{q,\ell,m} \right]_b \quad (3.4)$$

Since

$$L^2 - q^2 = [r \times (p - Ze\hat{A})]^2, \quad (3.5)$$

operating this on $\hat{Y}_{q,\ell,m}_a$ and $\hat{Y}_{q,\ell,m}_b$ give the same differential equation for $\hat{\Theta}_{q,\ell,m}$:

$$\begin{aligned} [\ell(\ell+1) - q^2] \hat{\Theta} = & -\frac{1}{\sin\theta} \partial_\theta [\sin\theta \partial_\phi \hat{\Theta}] - 2iq\gamma \sin\theta \partial_\theta \hat{\Theta} \\ & + \frac{1}{\sin^2\theta} [(m+q \sin\theta)^2 + q^2 \gamma^2 \sin^4\theta - 2iq\gamma \cos\theta \sin^2\theta] \hat{\Theta}. \end{aligned} \quad (3.6)$$

Letting $x = \cos\theta$, this becomes

$$\begin{aligned} [\ell(\ell+1) - q^2] \hat{\Theta} = & - (1-x^2) \hat{\Theta}'' + [2x + 2iq\gamma(1-x^2)] \hat{\Theta}' \\ & + \frac{1}{1-x^2} \left[(m+qx)^2 + q^2 \gamma^2 (1-x^2)^2 - 2iq\gamma x(1-x^2) \right] \hat{\Theta} \end{aligned} \quad (3.7)$$

where prime indicates differentiation with respect to x .

Now, equation of the form

$$y'' + P(x)y' + Q(x)y = 0 \quad (3.8)$$

can be reduced to the normal form [4]

$$u'' + q(x)u = 0 \quad (3.9)$$

where

$$y = u(x) \cdot v(x) \quad (3.10)$$

and

$$v(x) = \exp \left[-\frac{1}{2} \int P dx \right] \quad (3.11)$$

Applying this technique to Eq. (3.6), we have

$$v(x) = (1-x^2)^{\frac{1}{2}} e^{iq\gamma x} \quad (3.12)$$

$$u'' + \left\{ \frac{[1-(m+qx)^2]}{(1-x^2)^2} + \frac{[\ell(\ell+1)-q^2]}{(1-x^2)} \right\} u = 0. \quad (3.13)$$

Analysis shows that $x = \pm 1$, $x \rightarrow \infty$ are regular singularities of Eq. (3.13).

The solution of this is expressible in terms of the hypergeometric function [5]

$F(\alpha, \beta; \gamma; x)$

$$u(x) = (1+x)^a (1-x)^c {}_2F_1 \left[\begin{matrix} a+b+c, a+b'+c; 1+a-a' \end{matrix}; \frac{1+x}{2} \right] \quad (3.14)$$

$$a = \frac{1-(m-q)}{2} \quad a' = \frac{1+(m-q)}{2} \quad c = \frac{1-(m+q)}{2} \quad (3.15)$$

$$b = -(\ell+1) \quad b' = \ell$$

The solution can also be expressed in terms of the Jacobi polynomial [3]

$P_n^{\alpha, \beta}$. Let

$$n = \ell + n \quad \alpha = -(m+q) \quad \beta = -(m-q) \quad (3.16)$$

then

$$F \left[\begin{matrix} -n, n+\alpha+\beta+1; \beta+1 \end{matrix}; \frac{1+x}{2} \right] = (-1)^n \frac{n! \beta!}{(n+\beta)!} P_n^{\alpha, \beta}(x) \quad (3.17)$$

Hence,

$$u_{q, \ell, m}(x) = C(1-x)^{(1+\alpha)/2} (1+x)^{(1+\beta)/2} (-n)^n \frac{n! \beta!}{(n+\beta)!} P_n^{\alpha, \beta}(x) \quad (3.18)$$

and

$$\hat{\Theta}_{q,\ell,m}(x) = C e^{iq\gamma x} (1-x)^{\alpha/2} (1+x)^{\beta/2} (-1)^n \frac{n!\beta!}{(n+\beta)!} P_n^{\alpha,\beta}(x) \quad (3.19)$$

Grouping all the constants into $C_{q,\ell,m}$, we get

$$\hat{\Theta}_{q,\ell,m}(x) = C_{q,\ell,m} e^{iq\gamma x} (1-x)^{-(q+m)/2} (1+x)^{(q-m)/2} P_{\ell+m}^{-q-m,q-m}(x) \quad (3.20)$$

Imposing the normalization condition [6]

$$\int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} \left| \hat{Y}_{q,\ell,m} \right|^2 d\phi = 1 \quad (3.21)$$

we pinpoint the constant $C_{q,\ell,m}$

$$C_{q,\ell,m} = 2^m \left[\frac{(2\ell+1)}{4\pi} \frac{(\ell+m)! (\ell-m)!}{(\ell-q)! (\ell+q)!} \right]^{\frac{1}{2}} \quad (3.22)$$

which is exactly Eq. (1.17).

The result that $\hat{Y}_{q,\ell,m}$ are just a phase factor $e^{iq\gamma x}$ different from $Y_{q,\ell,m}$ of Wu and Yang can easily be seen if one reduces Eq. (23) of [1] to normal form. This can be seen right away if we let $\gamma = 0$ in Eqs. (3.11) and (3.12).

4. SCHRÖDINGER EQUATION

The relation

$$(\underline{p} - Ze\tilde{A})^2 = \frac{1}{r^2} \partial_r (r^2 \partial_r) + \frac{1}{r^2} [\tilde{r} \times (\underline{p} - Ze\tilde{A})]^2$$

holds as long as $\tilde{A}$ has no $\tilde{r}$ component. So, it is true for $\hat{\tilde{A}}$. Together with Eq. (3.5), this means letting

$$\psi = R(r) Y_{q,\ell,m}$$

in Eq. (1.8) or (1.9), one gets

$$\left[-\frac{1}{2mr^2} \partial_r (r^2 \partial_r) + \frac{\ell(\ell+1) - q^2}{2mr^2} + V - E \right] R(r) = 0.$$

As Ref. [1] mentions, this was solved by Tamm [7] for the case $V = 0$ and $E > 0$.

The solution is a Bessel function

$$R(r) = \frac{1}{\sqrt{kr}} J_{\mu}(kr)$$

$$\mu = \left[\left(\ell + \frac{1}{2} \right)^2 - q^2 \right]^{\frac{1}{2}} \quad k = \sqrt{2mE}$$

ACKNOWLEDGEMENT. The author acknowledges the encouragement and support of the Department of Physics of The Pennsylvania State University. She also thanks Professor Charles Fulton of the Mathematics Department for helpful discussions.

REFERENCES

1. Wu, T. T. and C. N. Yang, Dirac Monopole without Strings: Monopole Harmonics Nucl. Phys. B107 (1976) 365.
2. Wu, T. T. and C. N. Yang, Concepts of Nonintegrable Phase Factors and Global Formulation of Gauge Fields. Phys. Rev. D12 (1975) 3845.
3. Erdélyi, A. W. Magnus, F. Oberheltinger and F. Tricomi, Higher Transcendental Functions vol. 2 (McGraw-Hill, 1953) 168-170.
4. Simmons, G. F. Differential equations with applications and historical notes (McGraw-Hill, 1972) p. 119.
5. Whittaker E. T. and G. N. Watson, A Course in Modern Analysis (Cambridge University Press, 1965) 194-208.
6. Erdélyi, A. , W. Magnus, F. Oberheltinger and F. Tricomi, Tables of integral transforms vol. 2 (McGraw-Hill, 1953).
7. Tamm, Ig., Die verallgemeinerten Kugelfunktionen und die Wellenfunktionen eines Elektron im Felde eines Magnetpoles Z. Phys. 71 (1931) 141.