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SINGULAR PERTURBATION FOR NONLINEAR BOUNDARY-VALUE PROBLEMS

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ABSTRACT. Asymptotic solutions of a class of nonlinear boundary-value problems are studied. The problem is a model arising in nuclear energy distribution. For large values of the parameter, the differential equations are of the singular-perturbation type and approximations are constructed by the method of matched asymptotic expansions.

KEY WORDS AND PHRASES. *Asymptotic solutions of nonlinear boundary-value problems, large parameter, singular perturbation and method of matched asymptotic expansions.*

AMS (MOS) SUBJECT CLASSIFICATION (1970) CODES. *34B15, 34D15, 34E15.*

1. INTRODUCTION.

Many of the problems occuring in physics, engineering and applied mathematics contain a small parameter, and due to difficulties such as nonlinear equations, variable coefficients, the solution cannot be obtained exactly, see for example Cole (1) and Nayfeh (2). In this work, asymptotic solutions are obtained for non-

linear boundary value problems of the form

$$\frac{d^2 y}{d x^2} + r^2(x)y - f(x)\, y^n = 0 \ , \tag{1.1}$$

$$y(0) = a, \quad y(1) = b \ , \tag{1.2}$$

where $r^2(x)$ and $f(x)$ are positive functions, n is a positive integer ≥ 2, $a \geq 0$ and $b \geq 0$, by the method of matched asymptotic expansions.

The problem arises in connection with the distribution of the energy released in a nuclear power reactor as a result of a power excursion; $r^2(x)$ is the space-dependent perturbation in the neutron multiplication of a reactor, and (1.1) and (1.2) give the distribution of the energy release from the start of the perturbation till the neutron population again becomes zero, see Ergen (3). The case of zero boundary conditions and constant coefficients has been investigated in Canosa and Cole (4). It would be assumed that y is positive and bounded.

2. UPPER BOUND FOR THE MAXIMUM OF THE SOLUTION.

Let the maximum value of the solution occur at $x = c$, then

$$y(c) = M, \ y'(c) = 0, \ y''(c) < 0 \ . \tag{2.1}$$

From (1.1) and (2.1),

$$y''(c) = f(c)\, y^n(c) - r^2(c)\, y(c) < 0 \tag{2.2}$$

and so

$$y^{n-1}(c) < \frac{r^2(c)}{f(c)} \ ,$$

or

$$y(c) < \left[\frac{r^2(c)}{f(c)}\right]^{\frac{1}{n-1}}$$

An upper bound for the solution is given by

$$y(x) < \left[\text{maximum of } \frac{r^2(x)}{f(x)}\right]^{\frac{1}{n-1}}$$

Note that if both r^2 and f are constant, then (2.2) implies that y cannot have any relative minimum. For definitness, it would be assumed in general that

$$y'(0) \geq 0 \text{ and } y'(1) \leq 0 .$$

But the results can be easily modified for the other possibilities.

3. SINGULAR PERTURBATION PROBLEM.

Consider the asymptotic case

$$r^2(x) = \lambda\rho\ (x) , \tag{3.1}$$

where $\lambda \to \infty$, $\rho(x) = 0(1)$,
and both λ and $\rho(x)$ are positive.

Then (1.1) becomes

$$y'' + \lambda\rho(x)y - f(x)y^n = 0 . \tag{3.2}$$

Introducing the following new variable,

$$y = \lambda^{\frac{1}{n-1}} Y,\ Y = 0(1) , \tag{3.4}$$

equation (3.2) becomes

$$\varepsilon \frac{d^2 Y}{dx} + \rho(x)Y - f(x)\ Y^n = 0 , \tag{3.5}$$

where $\varepsilon = \frac{1}{\lambda} \to 0$.

Equation (3.5) is a singular perturbation equation, the asymptotic expansions of which and (1.2) will be studied in the remaining sections.

4. ASYMPTOTIC SOLUTION FOR $\rho(x) = 1$ AND n IN GENERAL.

The equation under consideration is

$$\varepsilon \frac{d^2 Y}{d x^2} + Y - f(x)\ Y^n = 0 . \tag{4.1}$$

(i) Outer Solution

Assuming the solution in the form of an asymptotic series

$$Y(x, \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i Y_i(x)$$

$$= Y_0(x) + \varepsilon Y_1(x) + \dots ,$$

and substituting it into (4.1), the functions $Y_i(x)$ can be determined recursively. The first two terms are given by

$$Y_0 = [\frac{1}{f(x)}]^{\frac{1}{n-1}}$$

and

$$Y_1 = \frac{Y''_0}{n - 1}$$

$$= \frac{1}{n - 1} \frac{d^2}{dx^2} [\frac{1}{f(x)}]^{\frac{1}{n-1}} .$$

(ii) Inner Solution

To study the solution near the boundary $x = 0$, let

$$\tilde{x} = \frac{x}{\varepsilon^{\frac{1}{2}}} ,$$

then (4.1) becomes

$$\frac{d^2Y}{dx^2} + Y - f(\varepsilon^{\frac{1}{2}} \tilde{x})Y^n = 0 . \tag{4.2}$$

The boundary-layer solution has the form

$$Y(\tilde{x}, \varepsilon) = g_0(\tilde{x}) + \alpha_1(\varepsilon) g_1(\tilde{x}) + \dots , \tag{4.3}$$

where $\alpha_1(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Let $f(\varepsilon^{\frac{1}{2}} \tilde{x})$ be expanded as

$$f(\varepsilon^{\frac{1}{2}} \tilde{x}) = f(0) + \alpha_1 (\varepsilon) f_\ell \tilde{x} + \dots . \tag{4.4}$$

Substitution of (4.3) and (4.4) into (4.2) leads to the following differential equation for $g_0(\tilde{x})$:

$$\frac{d^2 g_0}{d \tilde{x}^2} + g_0 - f(0)\, g_0^n = 0 ,$$

and so

$$\left(\frac{d\, g_0}{d\, \tilde{x}}\right)^2 + g_0^2 - \frac{1}{n+1} f(0)\, g_0^{n+1} = \text{constant}. \tag{4.5}$$

In matching with the outer solution, the constant in (4.5) can be obtained. If $f'(0) = 0$, then (4.5) becomes

$$\begin{aligned}
\left(\frac{d\, g_0}{d\, \tilde{x}}\right)^2 &= \frac{n-1}{n+1}\left[\frac{1}{f(0)}\right]^{\frac{2}{n-1}} - g_0^2 + \frac{2}{n+1} f(0)\, g_0^{n+1} \\
&= \frac{2\, f(0)}{n+1}\left\{ g_0^{n+1} - \frac{n+1}{2\, f(0)}\, g_0^2 + \frac{n-1}{2}\left[\frac{1}{f(0)}\right]^{\frac{n+1}{n-1}} \right\} \\
&= \frac{2\, f(0)}{n+1}\left\{ g_0 - \left[\frac{1}{f(0)}\right]^{\frac{1}{n-1}} \right\} \left\{ g_0^n + \left[\frac{1}{f(0)}\right]^{\frac{1}{n-1}} g_0^{n-1} \right. \\
&\quad + \left[\frac{1}{f(0)}\right]^{\frac{2}{n-1}} g_0^{n-2} + \dots + \left[\frac{1}{f(0)}\right]^{\frac{n-2}{n-1}} g_0^2 \\
&\quad \left. - \frac{n-1}{2\, f(0)}\, g_0 - \frac{n-1}{2}\left[\frac{1}{f(0)}\right]^{\frac{n}{n-1}} \right\} \\
&= \frac{2\, f(0)}{n+1}\left\{ g_0 - \left[\frac{1}{f(0)}\right]^{\frac{1}{n-1}} \right\}^2 P_{n-1}(g_0) ,
\end{aligned} \tag{4.6}$$

where $P_{n-1}(g_0)$ is a polynomial of degree $(n-1)$ in g_0. Note that $\left[\frac{1}{f(0)}\right]^{\frac{1}{n-1}}$ is a double root of the polynomial on the right hand side.

Therefore

$$\int_0^{g_0} \frac{ds}{\left\{ s - \left[\frac{1}{f(0)}\right]^{\frac{1}{n-1}} \right\} \left[P_{n-1}(s)\right]^{\frac{1}{2}}} = -\sqrt{\frac{2\, f(0)}{n+1}}\; \tilde{x} .$$

Similar can be obtained for the boundary layer at $x = 1$, with $g_0(\tilde{x})$ replaced by say $h_0(\tilde{\tilde{x}})$, where $\tilde{\tilde{x}} = (1 - x)/\varepsilon^{\frac{1}{2}}$, $f(0)$ by $f(1)$ and $f'(0)$ by $f'(1)$.

5. EXPLICIT ASYMPTOTIC SOLUTIONS FOR SPECIAL CASES AND DISCUSSION OF RESULTS.

There are two special cases in which explicit asymptotic solutions can be obtained. For $n = 2$, the first term in the outer expansion is given by

$$Y_0 = \frac{1}{f(x)} \quad .$$

Transforming back to the origional variable y, we see that, away from the boundaries, the solution is given asymptotically by $\frac{r^2}{f(x)}$. Equation (4.6) becomes

$$\left(\frac{d\, g_0}{d\, \tilde{x}}\right)^2 = \frac{1}{3f^2(0)} - g_0^2 + \frac{2}{3} f(0) g_0^3$$

$$= \frac{2\, f(0)}{3} \left(g_0 - \frac{1}{f(0)}\right)^2 \left(g_0 + \frac{1}{2f(0)}\right) .$$

Since $\left.\frac{d\, g_0}{d\, \tilde{x}}\right|_0 \geq 0$,

$$\frac{d\, g_0}{d\, \tilde{x}} = - \sqrt{\frac{2\, f(0)}{3}} \left(g_o - \frac{1}{f(0)}\right) \sqrt{g_0 + \frac{1}{2\, f(0)}}$$

or

$$\frac{d\, g_o}{\left(g_0 - \frac{1}{f(0)}\right) \sqrt{g_0 + \frac{1}{2\, f(0)}}} = - \sqrt{\frac{2\, f(0)}{3}}\, d\tilde{x} \, . \tag{5.1}$$

Integration of (5.1) and the boundary condition $g_0(0) = a$ lead to

$$\ln\left(\frac{\sqrt{3} - \sqrt{2\, f(0)\, g_0 + 1}}{\sqrt{3} + \sqrt{2\, f(0)\, g_0 + 1}}\right) = - \tilde{x} + \ln\left(\frac{\sqrt{3} - \sqrt{2f(0)a + 1}}{\sqrt{3} + \sqrt{2f(0)a + 1}}\right) , \left[a < \frac{1}{f(0)} \right]$$

therefore

$$g_0(\tilde{x}) = \frac{1}{f(0)} \frac{[1-af(0)]^2 - 4[1-af(0)][2+af(0)-\sqrt{6af(0)+3}]e^{-\tilde{x}} + [2+af(0)-\sqrt{6af(0)+3}]^2 e^{-2\tilde{x}}}{[1-af(0)]^2 + 2[1-af(0)][2+af(0)-\sqrt{6af(0)+3}]e^{-\tilde{x}} + [2+af(0)-\sqrt{6af(0)+3}]^2 e^{-2\tilde{x}}} \tag{5.2}$$

Near the boundary x = 1, the first term of the boundary-layer solution is given by

$$h_o(\tilde{\tilde{x}}) = \frac{1}{f(1)} \frac{[1-bf(1)]^2 - 4[1-bf(1)][2+bf(1)-\sqrt{6bf(1)+3}]e^{-\tilde{\tilde{x}}} + [2+bf(1)-\sqrt{6bf(1)+3}]^2 e^{-2\tilde{\tilde{x}}}}{[1-bf(1)]^2 + 2[1-bf(1)][2+bf(1)-\sqrt{6bf(1)+3}]e^{-\tilde{\tilde{x}}} + [2+bf(1)-\sqrt{6bf(1)+3}]^2 e^{-2\tilde{\tilde{x}}}} \tag{5.3}$$

Equations (5.2) and (5.3) show the exponential decay of the boundary solutions into the outer solution, and the symmetry of the solution about the domain center if f has such symmetry and a = b. The first term outer solution and (5.2) reduce to the ones given in Canose and Cole (4) when the coefficient $f(x) \equiv 1$ and the boundary conditions are zero.

When n = 3, the first term of the outer solution is

$$Y_0 = \frac{1}{\sqrt{f(x)}}$$

and so away from the boundaries, the solution is given asymptotically by $\frac{r}{\sqrt{f(x)}}$. Equation (4.6) now becomes

$$\left(\frac{d\, g_0}{d\, \tilde{x}}\right)^2 = \frac{1}{2\, f(0)} - g_0^2 + \frac{1}{2} f(0)\, g_0^4$$

$$= \frac{f(0)}{2} \left(g_o - \frac{1}{\sqrt{f(0)}}\right)^2 \left(g_o + \frac{1}{\sqrt{f(0)}}\right)^2 .$$

Therefore

$$\frac{d\, g_0}{d\, \tilde{x}} = -\sqrt{\frac{f(0)}{2}} \left(g_o - \frac{1}{\sqrt{f(0)}}\right) \left(g_0 + \frac{1}{\sqrt{f(0)}}\right)$$

and so

$$\frac{d\, g_0}{\left(g_0 - \frac{1}{\sqrt{f(0)}}\right)\left(g_0 + \frac{1}{\sqrt{f(0)}}\right)} = -\sqrt{\frac{f(0)}{2}}\, d\tilde{x} . \tag{5.4}$$

Integration of (5.4) and the boundary condition $g_0(0) = a$ lead to

$$\ell n \left(\frac{\frac{1}{\sqrt{f(0)}} - g_0}{\frac{1}{\sqrt{f(0)}} + g_0} \right) = -\sqrt{2}\,\tilde{x} + \ell n \left(\frac{\frac{1}{\sqrt{f(0)}} - a}{\frac{1}{\sqrt{f(0)}} + a} \right), \quad \left[a < \frac{1}{\sqrt{f(0)}} \right]$$

therefore

$$g_0(\tilde{x}) = \frac{1}{\sqrt{f(0)}} \; \frac{(1 + a\sqrt{f(0)})e^{\sqrt{2}\,\tilde{x}} - (1 - a\sqrt{f(0)})}{(1 + a\sqrt{f(0)})e^{\sqrt{2}\,\tilde{x}} + (1 - a\sqrt{f(0)})} \; . \tag{5.5}$$

Near $x = 1$, the first term of the boundary-layer solution is given by

$$h_0(\tilde{\tilde{x}}) = \frac{1}{\sqrt{f(1)}} \; \frac{(1 + b\sqrt{f(1)})e^{\sqrt{2}\,\tilde{x}} - (1 - b\sqrt{f(1)})}{(1 + b\sqrt{f(1)})e^{\sqrt{2}\,\tilde{x}} + (1 - b\sqrt{f(1)})} \; . \tag{5.6}$$

In this case, we see from (5.5) and (5.6), the exponential growth of the boundary-layer solutions into the outer solution, and again the symmetry of the solution about the domain center if f is symmetric and $a = b$. The first term outer solution and (5.5) reduce to the ones given in Canosa and Cole (4) when the coefficient $f(x) \equiv 1$ and the boundary conditions are zero.

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