RESEARCH NOTES

ON OSCULATORY INTERPOLATION BY TRIGONOMETRIC POLYNOMIALS

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ABSTRACT. A short and simple proof is given that osculatory interpolation by trigonometric polynomials is always possible.

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It is an elementary fact that if $2n + 1$ points θ_j with

$$-\pi \leq \theta_0 < \theta_1 < \dots < \theta_{2n} \leq \pi \quad (1)$$

are given, then there exists an element f of T_n , the class of trigonometric polynomials of degree $\leq n$;

$$f(e^{i\theta}) = \sum_{k=-n}^n a_k e^{ik\theta}, \quad (2)$$

so that $f(e^{i\theta_j}) = w_j$ for $j = 0, 1, \dots, 2n$, where $w_0, w_1, \dots, w_{2n}$ is any given sequence of complex numbers. A proof (in the real case) is given in Example 5 on page 38 of Davis' book [1], and on page 53, Problem 13 asks, somewhat enigmatically, "Discuss the possibility of osculatory trigonometric interpolation." In this note, we give a simple proof of a theorem that answers this problem and a bit more.

THEOREM. Given two sets of n complex numbers, $w_1, \dots, w_n$ and $w'_1, \dots, w'_n$, there exists a trigonometric polynomial f of the form 2), with $a_0 = 0$, so that $f(e^{i\theta_j}) = w_j$ and $f'(e^{i\theta_j}) = w'_j$ for $j = 1, 2, \dots, n$.

REMARK. The osculatory case is where all the $w_j = 0$. Our theorem amounts to letting the θ_j coalesce in pairs. Our proof depends on a trick that does not seem to cover more general kinds of coalescence, for which there is surely a corresponding result. The lemma we use to prove the theorem sheds a little light on the problem considered in [2-4] about the number of vanishing coefficients in the square of a polynomial.

PROOF OF THE THEOREM. Let T_n^0 be the subclass of T_n where $a_0 = 0$, so that T_n^0 is a vector space of dimension $2n$, and consider the $2n$ linear functionals consisting of point evaluations of elements of T_n^0 at the $e^{i\theta_j}$ and also of point evaluations of their first derivative at the $e^{i\theta_j}$, $j = 1, \dots, n$. By standard considerations of linear algebra it is enough to prove that these functionals are linearly independent, or equivalently, that if $f \in T_n^0$ and if $f(e^{i\theta_j}) = f'(e^{i\theta_j}) = 0$ for $j = 1, \dots, n$, then $f \equiv 0$. Let us suppose we have such an f .

Now writing $z = e^{i\theta}$ for z on the unit circle $T = \{|z| = 1\}$, we have

$$z^n f(z) = P(z) = \sum_{k=0}^{2n} b_k z^k,$$

where $P(z)$ is an algebraic polynomial of degree $2n$ whose coefficient b_n of degree n satisfies $b_n \neq 0$. Since the roots of P are at the distinct points $e^{i\theta_j}$ and are all double roots, we see that P is the square of a polynomial Q of degree n with n roots on the unit circle. The next lemma then settles the question.

LEMMA. Let Q be a polynomial of degree n with n roots on the unit circle. Then the middle coefficient b_n of $Q^2(z)$ does not vanish.

PROOF. Let

$$Q(z) = \prod_{j=1}^n (z - e^{i\theta_j})$$

so that

$$\frac{Q^2(z)}{z^n \prod_{j=1}^n (-e^{i\theta_j})} = \prod_{j=1}^n (z - e^{i\theta_j}) \prod_{j=1}^n (\frac{1}{z} - e^{-i\theta_j}).$$

Hence, on $\{|z| = 1\}$, we have

$$|Q(z)|^2 = A \frac{Q^2(z)}{z^n}, \quad (3)$$

where

$$A = \prod_{j=1}^n (-e^{-i\theta_j}).$$

Now integrate both sides of (3) around T with respect to the measure $d\theta/2\pi = (dz)/(2\pi iz)$ to get

$$Ab_n = \int_T |Q(z)|^2 \frac{dz}{2\pi iz} > 0.$$

Since $|A| = 1$, we see that $b_n \neq 0$.

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