ON THE OVERCONVERGENCE OF CERTAIN SERIES

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ABSTRACT. In this work, we consider certain class of exponential series with polynomial coefficients and study the properties of convergence of such series. Then we consider a subclass of this class and prove certain theorems on the overconvergence of such a series, which allow us to determine the conditions under which the boundary of the region of convergence of this series is a natural boundary for the function $f$ defined by this series.

KEY WORDS AND PHRASES. LC-Dirichletian element, L-Dirichletian element, Convergence, Overconvergence.

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1. INTRODUCTION.

Let us consider the following LC-dirichletian element

$$\{f\} : \sum_{n=1}^{\infty} P_n(x) \exp(-\lambda_n s), \quad (1.1)$$

where $P_n(s) = \sum_{j=0}^{m_n} a_{nj} s^j$, $a_{nj}$ are complex constants with $a_{nj} \neq 0$, $s = \sigma + i\tau$. 

\((\sigma, \tau) \in \mathbb{R}^2\) and \((\lambda_n)\) is a sequence of complex numbers such that \((|\lambda_n|)\) is a D-sequence. That is to say \((|\lambda_n|)\) is a sequence of positive real numbers satisfying
\[
0 < |\lambda_1| < |\lambda_2| < \ldots , \lim_{n \to \infty} |\lambda_n| = \infty. \tag{1.2}
\]

Let
\[
L = \lim \sup \left\{ \frac{\log n}{|\lambda_n|} / n \in \mathbb{N} - \{0\} \right\} \tag{1.3}
\]
\[
A_n = \max \left\{ |a_{nj}| / j \in (0, 1, \ldots, m_n) \right\} \tag{1.4}
\]
and
\[
\beta^* = \lim \sup \left\{ \frac{m_n}{|\lambda_n|} / n \in \mathbb{N} - \{0\} \right\}. \tag{1.5}
\]

Let \(\mathcal{E}_n\) be the set of points of \(\mathbb{C}\) which are zeros of \(P_n(s)\) and \(\mathcal{E} = \bigcup \mathcal{E}_n\). Let us denote by \(\mathcal{E}^d\) the derived set of \(\mathcal{E}\) and
\[
\mathcal{E}_\infty = \left\{ s \in \mathbb{C} \mid \exists s \in (n_j) P_n(s) = 0 \right\} \text{ where } (n_j) \text{ is an infinite subsequence of } \mathbb{N} - \{0\}
\]
depending on \(s\); let \(\mathcal{E}^* = \mathcal{E}^d \cup \mathcal{E}_\infty\). \(\mathcal{E}^*\) is a closed set. Let us suppose that \(\mathbb{C} - \mathcal{E}^*\) is non empty. We put
\[
\forall s \in \mathbb{C} - \mathcal{E}^*, \quad \delta(n, s) = -\frac{\log |P_n(s) \exp(-\lambda_n s)|}{|\lambda_n|}, \text{ for sufficiently large } n, \tag{1.6}
\]
\[
\delta_\ast(s) = \lim \inf \left\{ \delta_n(s) / n \in \mathbb{N} - \{0\} \right\} \tag{1.7}
\]
\[
\forall \varepsilon \in \mathbb{R}, \quad \mathcal{B}_\ast = \left\{ s \in \mathbb{C} - \mathcal{E}^* / \delta_\ast(s) > \varepsilon \right\}. \tag{1.8}
\]

In this paper, using a technique similar to that used by M. Blamert and J. Simeon [2], we prove two lemmas for a LC-dirichletian element which enable us to discuss the properties of absolute convergence and uniform convergence for (1.1) in \(\mathbb{C} - \mathcal{E}^*\) exclusively. Then we prove Jentzsch's theorem for a L-dirichletian element that is for element of the type (1.1) where \(\lambda_n\) are positive real numbers satisfying (1.2) \((\lambda_n)\) is a D-sequence and a theorem on the overconvergence for a L-dirichletian element.

2. MAIN RESULTS.

DEFINITION. - It is said that a function is sub-lipschitzian on an open set, if it is lipschitzian on each compact subset of that open set.
**Lemma 1.** Let \( \kappa \) be any compact subset of \( \mathbb{C} \). Then the following assertions are true.

1. \( \forall \exists \forall \) the function \( \kappa \ni s \rightarrow \delta(n, s) \) is Lipschitzian.

2. If \( \beta^* < \infty \), and if there exists a \( s_0 \in \mathbb{C} - \beta^* \) such that \( |\delta^*(s_0)| < \infty \), then the function \( \delta^* \) is sub-Lipschitzian on \( \mathbb{C} - \beta^* \).

**Proof.** Let \( \forall \exists \forall \) \( \kappa \ni \epsilon = \text{dist}(\kappa, \beta^*) \). Then it is easy to see that.

\[
\forall \exists \forall \forall \{ j \in \{1, 2, \ldots, m_n\} = \alpha_{n_j} \notin d_s, \epsilon \},
\]

where \( d_{s, \epsilon} \) is the open disc centred at \( s \) and of radius \( \epsilon \) and \( \{ \alpha_{n_j} \} \), \( j \in \{1, 2, \ldots, m_n\} \), is the sequence of zeros of \( P_n(s) \) (with its order of multiplicity is taken into account). More precisely let us show that.

\[
\forall \exists \forall \forall \forall \{ j \in \{1, 2, \ldots, m_n\} = \alpha_{n_j} \notin d_s, \epsilon \},
\]

Let \( G_\epsilon = \bigcup \{ d_{s, \epsilon} \} \). It is evident that \( \overline{G}_\epsilon \) the closure of \( G_\epsilon \) is a compact subset of \( \mathbb{C} - \beta^* \). Let \( \epsilon' \ni \bigcap \epsilon \notin \bigcap \epsilon \). The set of discs \( d_{s, \epsilon'} \) indexed by \( s \) on \( \overline{G}_\epsilon \) is an open covering of \( \overline{G}_\epsilon \). Hence we have a finite subcovering.

\[
\exists \bigcup_{j=1}^{k} d_{s_j, \epsilon'} = \overline{G}_\epsilon.
\]

Let \( s \in \kappa \) and \( s' \in d_{s, \epsilon} \); hence \( s' \in G_\epsilon \). Then \( s' \in \bigcup_{j=1}^{k} d_{s_j, \epsilon'} \) which implies that \( \exists s' \in d_{s_j, \epsilon'} \). Now

\[
\forall \exists \forall \forall \{ j \in \{1, \ldots, k\} \}
\]

and hence

\[
\forall \exists \forall \forall \{ j \in \{1, \ldots, k\} \} \quad P_n(s) \neq 0,
\]

which gives \( \forall \exists \forall \forall \{ j \in \{1, \ldots, k\} \} \quad P_n(s') \neq 0 \). As \( s \) is arbitrary on \( \kappa \) and \( s' \) is arbitrary on \( d_{s, \epsilon} \) we have
From which we have

$$\forall \epsilon \in ]0, \epsilon_{K}[^{\mathbb{R}} \quad \forall n, n' \ni \exists j(1, \ldots, k) = \alpha_{nj} \not= s_{j}, \epsilon \} .$$

Under the above conditions related to \( n, s \) and \( s' \) with \( s \neq s' \),

$$|\delta(n, s) - \delta(n, s')| \leq |s-s'| + \frac{1}{|\lambda_{n}|} \sum_{j=1}^{m_{n}} \log \left\{ 1 + \frac{|s-s'|}{\epsilon} \right\}$$

$$\leq |s-s'| + \frac{|s-s'|}{|\lambda_{n}|} \sum_{j=1}^{m_{n}} \log \left( 1 + \frac{|s-s'|}{\epsilon} \right)$$

$$\leq |s-s'| + \frac{|s-s'|}{|\lambda_{n}|} \sup_{x > 0} \left\{ \frac{\log(1+x)}{x} \right\} ;$$

as \( \sup_{x > 0} \left\{ \frac{\log(1+x)}{x} \right\} = 1 \), \( |\delta(n, s) - \delta(n, s')| \leq |s-s'| \left\{ 1 + \frac{m_{n}}{\epsilon |\lambda_{n}|} \right\} \). Putting

$$\mu_{n} = 1 + \frac{m_{n}}{\epsilon |\lambda_{n}|} ,$$

$$\forall \epsilon \in ]0, \epsilon_{K}[^{\mathbb{R}} \quad \forall (s, s') \in \mathbb{K} \times \mathbb{K}$$

which proves the first part of the lemma.

Now let \( \mu_{n} = \lim_{n \to \infty} \mu_{n} = 1 + \beta_{n}/\epsilon \) with \( \epsilon \in ]0, \epsilon_{K}[^{\mathbb{R}} \); as \( \exists s_{0} \in C-\epsilon^{*} \delta_{*}(s_{0}) < \infty \)

$$\forall \epsilon \in ]0, \epsilon_{K}[^{\mathbb{R}} \quad \forall (s, s') \in \mathbb{K} \times \mathbb{K}$$

and

$$\forall (s, s') \in \mathbb{K} \times \mathbb{K}$$

where

$$\mu^{*}_{K} = \inf_{\epsilon \in ]0, \epsilon_{K}[^{\mathbb{R}}} \left\{ \mu^{*}_{n}/\epsilon \right\} = 1 + \beta^{*}_{\epsilon_{K}} .$$

Hence

$$\forall (s, s') \in \mathbb{K} \times \mathbb{K}$$

which completes the proof of the lemma.

Under the condition (2) of Lemma 1, \( \delta_{*} \) is continuous on \( C-\epsilon^{*} \).
which implies that \( \mathcal{B}_{\alpha} \) is an open subset of \( \mathbb{C} - \varepsilon^* \); but \( \mathcal{B}_{\alpha} \) can have several connected components.

**Lemma 2.** When \( \beta^* < \infty \), then

\[
\forall \alpha \in \mathbb{R} \left\{ \mathcal{B}_{\alpha} \neq \emptyset \Rightarrow \forall \beta' \in \mathcal{B}_{\alpha} \mathcal{B}_{\alpha}^* \right. \forall n, n \geq n' \forall s \in \mathcal{K} \left| \prod_n \exp(-\lambda_n) < \exp(-\lambda_n^*(\alpha - \beta')) \right\}. 
\]

**Proof.** Let \( \alpha \in \mathbb{R} \) such that \( \mathcal{B}_{\alpha} \neq \emptyset \) (otherwise the lemma is trivial) and let \( \mathcal{K} \) be a compact subset of \( \mathcal{B}_{\alpha} \). We can easily see that

\[
\forall s \in \mathbb{C} - \varepsilon^* \quad s \in \mathcal{K} \quad \exists \forall \varepsilon > 0 \quad \prod_n \exp(-\lambda_n) < \exp(-\lambda_n^*(\alpha - \beta')) 
\]

where \( d_{s, \varepsilon} \) is the closed disc centred at \( s \) and of radius \( \varepsilon \). Hence

\[
\forall s \in \mathbb{C} - \varepsilon^* \quad \exists \forall \varepsilon > 0 \quad \prod_n \exp(-\lambda_n) < \exp(-\lambda_n^*(\alpha - \beta')) 
\]

Let us consider the compact subset \( \mathcal{K} = \bigcup_{s \in \mathcal{K}} d_{s, \varepsilon} \), of \( \mathbb{C} - \varepsilon^* \). As

\[
\forall s \in \mathbb{C} - \varepsilon^* \quad s \in \mathcal{K} \quad \exists \forall \varepsilon > 0 \quad \prod_n \exp(-\lambda_n) < \exp(-\lambda_n^*(\alpha - \beta')) 
\]

and hence

\[
\forall s \in \mathbb{C} - \varepsilon^* \quad \exists \forall \varepsilon > 0 \quad \prod_n \exp(-\lambda_n) < \exp(-\lambda_n^*(\alpha - \beta')) 
\]

Since \( \mathcal{K} \) is a compact subset of \( \mathcal{B}_{\alpha} \), \( \forall s \in \mathcal{K} \quad \exists n \neq n' \quad \delta(n, s') > \alpha \); finally we have

\[
\forall s \in \mathcal{K} \quad \exists n \neq n' \quad \delta(n, s') > \alpha \quad (2.1)
\]
where $\epsilon$ is arbitrary in $]0, \epsilon[$. The set of discs $d_{s, \epsilon}$ indexed by $s$ on $\kappa$ is an open covering for $\kappa$ and hence $\exists \bigcup_{s=1}^{k} d_{s, \epsilon} \supset \kappa$. Further we have $\forall \exists s \in d_{s, \epsilon}$.

Using (2.1) for the particular pair $(s_j, s)$, we have

$$
\forall \beta > \beta^* \epsilon \in ]0, \text{dist}(s_j, s)\left[ n(=n_{s_j, \epsilon}) n \in \kappa^n \right].
$$

Let $n'' = \text{Max}\left\{ n_{s_j, \beta, \epsilon'} \mid j \in (1..k) \right\}$ and as $|s - s_j| < \epsilon$, we have

$$
\forall \exists \delta(n, s) > \alpha - |s - s_j|\left(1 + \frac{\beta'}{\epsilon} \right).
$$

Choosing $\epsilon = \epsilon' < \frac{\epsilon}{3}$ we have $\frac{\text{dist}(k, \epsilon')} {2} < \text{dist}(c^*, \kappa, \epsilon)$ and

$$
\forall \beta > \beta^* \epsilon \in ]0, \frac{\epsilon}{2}[ \text{ n'' nzn''}.
$$

where $s$ is any arbitrary point of $\kappa$ and $n''$ does not depend on $s$. Hence

$$
\forall \beta > \beta^* \epsilon \in ]0, \frac{\epsilon}{2}[ \text{ n'' nzn''} s \in \kappa.
$$

As $\beta'$ is arbitrary and strictly greater than $\beta^*$, we have

$$
\forall \beta > \beta^* \in ]0, \frac{\epsilon}{2}[ \text{ n'' nzn''} s \in \kappa.
$$

and hence

$$
\forall \beta > \beta^* \in ]0, \frac{\epsilon}{2}[ \text{ n'' nzn''} s \in \kappa.
$$

THEOREM 1. - When $\beta^* < \infty$, $L < \infty$, the LC-dirichletian element $\{f\}$ converges absolutely on $\beta^*, L+\beta^*$ and uniformly on any compact subset of $\beta^*, L+\beta^*$.

PROOF. Let us suppose that $\beta^*, L+\beta^*$ is non empty. Let $\kappa_0$ be a compact subset of $\beta^*, L+\beta^*$. We know that $\exists \kappa_0 \subset \beta^*$. Let
\( \beta' \in ]\beta^*, \alpha - L[ \). From Lemma 2 we have,

\[ \exists \forall \forall' \sum_{n, n' \geq n'}|P_n(s)\exp(-\lambda_n s)| < \exp(-|\lambda_n| (\alpha - \beta')) \]

where \( \alpha - \beta' > L \). Hence

\[ \sum_{n, n' \geq n'}|P_n(s)\exp(-\lambda_n s)| < \exp(-|\lambda_n| (\alpha - \beta')) \]

and the series on the right hand side is convergent which proves that \( f \) converges absolutely and uniformly on \( \kappa_0 \). Since \( \kappa_0 \) is any arbitrary compact subset of \( \mathcal{D}_{*, L + \beta^*} \), \( f \) converges uniformly on any compact subset of \( \mathcal{D}_{*, L + \beta^*} \) and absolutely on \( \mathcal{D}_{*, L + \beta^*} \).

**REMARK 1.** By the following method, we obtain a bigger set of absolute convergence for \( f \). Let \( \mathcal{D}_{L} \) be supposed to be non-empty and \( L < \infty \).

Then \( \forall \exists \delta_*(s) > L + \varepsilon_s ; \exists \forall (n, s) > L + \varepsilon_s \) and

\[ n' s \geq n' \]

\[ \forall n' s \geq n' \]

\[ \sum_{n, n' \geq n'}|P_n(s)\exp(-\lambda_n s)| < \sum_{n, n' \geq n'}\exp(-|\lambda_n| (\alpha + \beta')) \]

and as the series on the right hand side converges, the series (1.1) converges absolutely on \( \mathcal{D}_{L} \). In this result, we have no restriction on \( \beta^* \).

**REMARK 2.** \( f \) diverges on \( \mathcal{C} - \mathcal{C}^* - \mathcal{D}_{\Omega} \). If \( s \in \mathcal{C} - \mathcal{C}^* - \mathcal{D}_{\Omega} \), then

\[ \delta_*(s) < 0 \] and \( \exists \delta_*(s) < -\alpha \). Hence \( \forall \exists \exists \delta(n, s) < -\alpha \)

\[ n' s \geq n' \]

\[ s \in \mathcal{C} - \mathcal{C}^* - \mathcal{D}_{\Omega} \]

where \( (n, s) \) is an infinite subsequence of \( \mathbb{N} - \{0\} \). Therefore

\[ |P_n(s)\exp(-\lambda_n s)| > \exp(\alpha|\lambda_n|) > 1 \]

and which shows that \( f \) diverges on \( \mathcal{C} - \mathcal{C}^* - \mathcal{D}_{\Omega} \). When \( L = 0 \), we have convergence of the series (1.1) in \( \mathcal{D}_{\Omega} \subset \mathcal{C} - \mathcal{C}^* \) and divergence in \( \mathcal{C} - \mathcal{C}^* - \mathcal{D}_{\Omega} \). We do not discuss the property of convergence of the series in \( \mathcal{C}^* \).

From here onwards we consider a \( L \)-dirichletian element,
\[
\{f\} : \sum_{n=1}^{\infty} A_n \exp(-\lambda_n s) \tag{2.2}
\]
where \(\{\lambda_n\}\) is a D-sequence (here \(\lambda_n\) are positive real numbers).

**DEFINITION.** It is said that a D-sequence \(\{\lambda_n\}\) is of the type \((\Lambda)\) if the following conditions are satisfied:

1. The Dirichlet series \(\sum_{j=1}^{\infty} \exp(-\lambda_s)\) converges on \(P = \{s \in \mathbb{C} \mid \sigma > 0\}\). (this gives that \(\sum_{j=1}^{\infty} \exp(-s(\lambda_n - \lambda_{jn}))\) converges on \(P\). Let \(\theta_n(s)\) be its sum at the point \(s\));

2. \(\forall \eta > 0\) the sequence of functions \(\theta_n(s)\) where \(\theta_n : P \ni s \rightarrow \theta_n(s)\) is bounded on \(P\);

3. For every \(\eta > 0\) the sequence of functions \(\theta_n^*(s)\) where \(\theta_n^* : P \ni s \rightarrow \sum_{j=1}^{\infty} \exp(-s(\lambda_n - \lambda_j))\) is bounded on \(P\).

**EXAMPLE.** – If \(\{\lambda_n\}\) is a D-sequence and \(\exists \mu > 0\) such that \(\inf(\lambda_{n+1} - \lambda_n) = \mu\), then it is easy to see that \(\{\lambda_n\}\) is of the type \((\Lambda)\).

If the D-sequence \(\{\lambda_n\}\) is of the type \((\Lambda)\), then we can easily show that \(L = 0\).

Now let us prove Jentzsch’s theorem for L-dirichletian element. This theorem for Dirichlet series with complex exponents was proved by T.M. Gallie [3]. First let us consider the associated Dirichlet series of \(\{f\}\).

\[\{f_A\} : \sum_{n=1}^{\infty} A_n \exp(-\lambda_n s)\]

where \(A_n\) is defined by (1.4). Let

\[\sigma_p^f = \text{Inf}\{\sigma \in \mathbb{R} \mid \lim_n |A_n \exp(-\lambda_n s)| = 0, n \to \infty\}\]

be the abscisse of pseudo convergence of \(\{f_A\}\). Then we know that

\[\sigma_p^f = \lim_{n \to \infty} \sup \left\{ \frac{\log A_n}{\lambda_n} \right\} ;
\]

when \(L = 0\), \(\sigma_p^f\) is the same as \(\sigma_c^f\), the abscisse of convergence of \(\{f_A\}\).
Let \( n \) and \( n' \) be two natural numbers such that \( n' \geq n \). Let \( E_{n,n'} \) denote the set, indexed by \((n,n')\), of points of \( \mathbb{C} \) which are zeros of the LC-dirichletian polynomial

\[
S_{n,n'}(S) = \sum_{j=n}^{n'} P_j(s) \exp(-s\lambda_j);
\]

let \( E \) denote the union of all sets \( E_{n,n'} \), corresponding to all pairs \((n,n')\) and \( E_{\infty} \) be the set formed by the points which are zeros for an infinity of polynomials \( S_{n,n'}(s) \). Let us put \( E^* = E^d \cup E_{\infty} \) where \( E^d \) is the derived set of \( E \). \( E^* \) is a closed subset of \( \mathbb{C} \). It is evident that \( E \supset \mathcal{E} \) and \( E_{\infty} \supset \mathcal{E}_{\infty} \) and hence \( E^* \supset \mathcal{E}^* \). We suppose in what follows that \( \mathbb{C} - E^* \neq \emptyset \) (which implies \( \mathbb{C} - \mathcal{E}^* \neq \emptyset \)). Then we have

**THEOREM 2.** When the D-sequence \( \left( \lambda_n \right) \) is of the type \( \frac{f_A}{n} \), \( \sigma_c < \infty \) and \( \beta^* < \infty \), then we have \( \left( \mathfrak{Fr}(\mathcal{E}_{\infty}) \cap \mathbb{C} - \mathcal{E}^* \right) \subset E^* \).

**PROOF.** Let us suppose that the theorem is not true. Then there exists a point \( b \in \left( \mathfrak{Fr}(\mathcal{E}_{\infty}) \cap \mathbb{C} - \mathcal{E}^* \right) \) and a disc \( d(b,\rho) \) centred at \( b \) of radius \( \rho > 0 \), included in \( \mathbb{C} - \mathcal{E}^* \) such that

\[
\exists n_0 \quad \forall n' \geq n_0 \quad \exists \in d(b,\rho) \quad S_{n,n'}(s) \neq 0.
\]

We have \( |P_n(s)\exp(-\lambda_n s)| \leq A_n(1+|s|)^{m_n} |\exp(-\lambda_n s)| \) and

\[
\forall \exists \in d(b,\rho) \quad (m/\lambda_n) < \beta' \quad \text{Let us take a certain} \quad \beta' > \beta^* \quad \text{and put} \quad \beta'^* n_0' \quad \text{such that} \quad \omega = \beta' \Log[1+\sup\{ |s|/s \in d(b,\rho) \}] - \inf\{ \sigma/s \in d(b,\rho) \} \quad \text{and hence}
\]

\[
\forall \exists \in d(b,\rho) \quad |P_n(s)\exp(-\lambda_n s)| < A_n \exp(\omega \lambda_n). \quad \text{From the definition of} \quad \frac{f_A}{n} \quad \text{we have}
\]

\[
\forall \exists \in d(b,\rho) \quad A_n < \exp(\sigma' \lambda_n). \quad \text{Hence putting} \quad n_1 = \max(n_0',n_0''), \quad \sigma' > \sigma_c \quad \text{and} \quad n_2 \quad \text{we get}
\]

\[
\forall \exists \in d(b,\rho) \quad |P_n(s)\exp(-\lambda_n s)| < \exp((\omega+\sigma') \lambda_n) \quad \text{with} \quad n_1 \quad \text{and} \quad n_2.
\]

Let \( S_{n,n'}(s) = \sum_{j=1}^{n} P_j(s) \exp(-\lambda_j s) \) and \( \forall s \in d(b,\rho) \quad T_{n,n}(s) = (S_{n,n}(s))^{1/\lambda_n}; \)

\[
[S_{n,n}(s)]^{1/\lambda_n} \quad \text{is defined to be equal to} \quad \exp((1/\lambda_n) \Log S_{n,n}(s)) \quad \text{where} \]

Im \log S_{n_1,n}(s) \in ]-\pi,\pi]. For each integer \( n \geq n_1 \) the function

\[ T_{n_1,n} : d(b,\rho) \ni s \to T_{n_1,n}(s) \text{ is holomorphic on } d(b,\rho). \]

We have

\[ \forall s \in d(b,\rho), \mid T_{n_1,n}(s) \mid = \left( \sum_{j=1}^{n} P_j(s) \exp(-\lambda_j s) \right)^{1/\lambda_n} \leq \exp(\log n) \exp(\log n). \]

Since \( (\lambda_n) \) is of the type \( (\lambda) \) which implies \( L = 0 \), we have \( \lim_{n \to \infty} \exp(-\log n) = 1. \)

Hence the sequence of functions \( (T_{n_1,n})_{n \geq n_1} \), is bounded and hence normal on \( d(b,\rho) \).

Let \( \kappa \) be a compact subset of \( d(b,\rho) \) such that \( \text{Int } \kappa \cap \partial \neq \emptyset \). From any extracted subsequence of \( (T_{n_1,n}) \) we can extract a subsequence which converges uniformly on \( \kappa \) and the limit function is holomorphic on the \text{Int } \kappa.

Let \( \kappa_1 \) be a compact subset of \( d(b,\rho) \) such that \( \text{Int } \kappa \cap \text{Int } \kappa_1 \neq \emptyset \). Then we have \( \forall s \in \kappa_1, \lim_{n \to \infty} T_{n_1,n}(s) = 1. \)

Now \( \kappa \cup \kappa_1 \) is a compact subset of \( d(b,\rho) \). Then the subsequence extracted from the arbitrarily extracted subsequence of \( (T_{n_1,n}) \) converges uniformly on \( \kappa \cup \kappa_1 \) to a limit function holomorphic in \( \text{Int } (\kappa \cup \kappa_1) \) and continuous on the boundary of \( \kappa \cup \kappa_1 \) and takes the value one at each point of \( \kappa_1 \). Hence the limit function takes the value one at each point of \( \kappa \cup \kappa_1 \). This results that the sequence \( (T_{n_1,n}) \) converges to the same limit function on \( \kappa \cup \kappa_1 \).

As \( \kappa \) is any arbitrary compact subset of \( d(b,\rho) \) and \( \kappa_1 \) is any arbitrary compact subset of \( d(b,\rho) \cap \partial \neq \emptyset \) such that \( \text{Int } \kappa \cap \text{Int } \kappa_1 \neq \emptyset \), we have

\[ \forall s \in d(b,\rho), \lim_{n \to \infty} T_{n_1,n}(s) = 1. \]

Let \( s_0 \in d(b,\rho) \cap (\mathbb{C} - \partial \neq \emptyset) \). Then

\[ \forall \epsilon > 0, \exists n_1 = n_1(s_0,\epsilon) \geq n_1, n \geq n_1 \]

and hence
which gives
\[
\frac{\log |P_n(s_0) \exp(-\lambda_n s_0)|}{\lambda_n} > -\frac{\log 2}{\lambda_n} - \log(1+\epsilon);
\]
\[
\delta_*(s_0) > 0 \text{ as } \epsilon \text{ is arbitrary. Hence we arrive at a contradiction that } s_0 \notin \mathcal{B}_{\delta_0} \cap \mathbb{C}-\mathbb{E}^* \text{ which establishes the result.}
\]

Finally, let us prove a theorem on the overconvergence of \( \{f\} \) defined by (7). Before proving the theorem let us note that

REMARK 3. Let \( \Delta \) be any compact subset of \( \mathbb{C}-\mathbb{E}^* \) and \( (\lambda_n) \) be a D-sequence of the type (\( \Lambda \)). We have
\[
|P_n(s) \exp(-\lambda_n s)| \leq A_n (1+|s|)^{\lambda_n} \exp(-\sigma_n).
\]
If \( s \in \Delta \), then
\[
|P_n(s) \exp(-\lambda_n s)| \leq A_n (1+m_\Delta)^{\lambda_n} \exp(\sigma_\Delta \lambda_n)
\]
where \( m_\Delta = \sup \{|s|/s \in \Delta\} \). As \( \Delta \) is a compact set, \( m_\Delta \) is finite; for sufficiently large \( n \) we have
\[
\frac{\log |P_n(s) \exp(-\lambda_n s)|}{\lambda_n} \leq \frac{\log A_n}{\lambda_n} + \frac{m_\Delta}{\lambda_n} \log(1+m_\Delta) + m_\Delta;
\]
\[
\delta_*(s) = -\frac{f_A}{\alpha_0 - \beta* \log(1+m_\Delta) - m_\Delta}.
\]
Hence \( \forall \Delta \subset \mathcal{B}_{*, \alpha_0 - \epsilon} \) with \( \alpha_0 = -\frac{f_A}{\alpha_0 - \beta* \log(1+m_\Delta) - m_\Delta} \). If
\[
\beta* < \frac{-\sigma_c - m_\Delta}{1+\log(1+m_\Delta)}, \text{ we have } \Delta \subset \mathcal{B}_{* \beta^*}.
\]

THEOREM 3. - When \( (\lambda_n) \) is a D-sequence of the type (\( \Lambda \)), \( \beta* \in \mathbb{R} \) and \( \mathcal{B}_{* \beta^*} \neq \phi \) if there exist an infinite subsequence \( (n_\nu) \), \( \nu \in \mathbb{N} \), of \( \mathbb{N}-\{0\} \)
and a sequence of strictly positive numbers \( (\theta_\nu) \) such that
\[
\lim_{\nu \to \infty} \theta_\nu = +\infty
\]
and
\[
\forall \nu \in \mathbb{N}, \lambda_{n \nu + 1} > (1+\theta_\nu)\lambda_{n \nu}
\] (2.3)
then the sequence \( \{S_n(s)\} \), \( n \in \mathbb{N} \), where \( S_n(s) = \sum_{j=1}^{\infty} P_j(s) \exp(-\lambda_j s) \), converges at each point \( s \) of any open simply connected subset (whose intersection with \( \mathbb{C} - \mathbb{C}^* \) is non empty) of an open set included in \( \mathbb{C} - \mathbb{C}^* \) in which the function \( f \) defined by \( \{f\} \) is holomorphic.

**PROOF.** Let us choose 3 bounded domains \( \Delta_1, \Delta_2 \) and \( \Delta_3 \) in the following manner: \( \Delta_1 \subset \Delta_2 \), \( \Delta_2 \subset \Delta_3 \), \( \Delta_3 \subset \mathbb{C} - \mathbb{C}^* \), \( \Delta_1 \subset \mathbb{C}^* \) and \( \Delta_3 \) is included in an open subset of \( \mathbb{C} - \mathbb{C}^* \) in which the function \( f \) defined by \( \{f\} \) is holomorphic. Further let \( \text{Fr}(\Delta_1) \), \( \text{Fr}(\Delta_2) \) and \( \text{Fr}(\Delta_3) \) satisfy a condition of Hadamard's type, namely

\[
\exists \quad \log M_2 \leq b \log M_1 + (1-b) \log M_3
\]

where \( b \in [0, 1] \)

\[M_i = \max_{s \in \text{Fr}(\Delta_i)} |f(s)| \]

It is easy to see that \( \exists \Delta \subset \mathbb{C} - \mathbb{C}^* \). Let us consider the set \( \Delta \) which is non empty and is an interval. Let \( a_{\Delta} = \sup I \). Then

\[a_{\Delta} > b^* \quad \text{and} \quad \forall \quad \delta^* < a_{\Delta} - \epsilon \subset \Delta \].

We can easily show that,

\[a_{\Delta} = \inf \{\delta^*(s) | s \in \Delta \} \]

which implies that \( a_{\Delta} \) is a finite number. Hence from lemma 2,

\[
\exists \quad \forall \quad \forall \quad \forall \quad [P_n(s) \exp(-\lambda_j s)] < \exp(-\lambda_n (a_{\Delta} - \beta')) \]

Since \( (\lambda_n) \) is a D-sequence of the type \((\lambda)\) and \( a_{\Delta} - \beta' > 0 \), there exists a finite number strictly positive \( B(\beta') \) such that

\[\forall \quad \forall \quad \sum_{j=n+1}^\infty |\exp(-\lambda_j s)| < B(\beta') \]

then we have for each \( n \geq n_1 \)
\[ \sum_{j=n+1}^{\infty} \left| P_j(s) \exp(-\lambda_j s) \right| < B(\beta') \exp\{-\lambda_{n+1}(\alpha_{\Delta_3} - \beta')\}. \]  

(2.4)

Now let \( I_2 = \{ \alpha \in \mathbb{R} \mid \beta_3 \geq \alpha \} \). We have

\[ \forall \, \exists \, \forall \, \delta(n,s) \geq \frac{-\log A_n}{\lambda_n} - \frac{m_n}{\lambda_n} \log (1+|s|) + \sigma. \]

Let \( m_{\Delta_3} = \sup \{|s|/s \in \Delta_3\} \). Then

\[ \forall \, \delta(n,s) \geq \frac{-\log A_n}{\lambda_n} - \frac{m_n}{\lambda_n} \log (1+m_{\Delta_3}) - m_{\Delta_3}, \]

which shows that \( \Delta_3 \subset \beta_3 \), with \( \alpha < -\sigma \frac{A_3}{c} - \beta^* \log (1+m_{\Delta_3}) - m_{\Delta_3} \), and hence \( I_2 \neq \emptyset \) and is an interval in \( \mathbb{R} \). Let \( \alpha_{\Delta_3} = \sup I_2 \). Then \( \forall \, \beta_3 < \alpha_{\Delta_3} - \varepsilon \in \Delta_3 \).

We can easily show that \( \alpha_{\Delta_3} = \inf \{ \delta_+(s) \mid s \in \Delta_3 \} \), which implies that \( \alpha_{\Delta_3} \) is a finite number. Once again, from lemma 2, we get

\[ \forall \, \exists \, \forall \, |P_n(s)\exp(-\lambda_n s)| < \exp\{-\lambda_n(\alpha_{\Delta_3} - \beta')\} \]

which gives

\[ \forall \, \sum_{s \in \Delta_3} |P_j(s)\exp(-\lambda_j s)| = \sum_{j=1}^{n_2-1} |P_j(s)\exp(-\lambda_j s)| + \sum_{j=n_2}^{n-1} |P_j(s)\exp(-\lambda_j s)| \]

\[ \leq \max \left\{ \sum_{j=1}^{n_2-1} |P_j(s)\exp(-\lambda_j s)| / s \in \Delta_3 \right\} + \sum_{j=n_2}^{n} \exp(-\lambda_j(\alpha_{\Delta_3} - \beta')) \].

Let us choose \( \beta' > \beta^* \) such that \( \alpha_{\Delta_3} - \beta' \neq 0 \). Now we examine the two cases.

**Case 1.** If \( \alpha_{\Delta_3} - \beta' > 0 \), then

\[ \sum_{j=n_2}^{n} \exp(-\lambda_j(\alpha_{\Delta_3} - \beta')) = \exp(\lambda_n(\alpha_{\Delta_3} - \beta')) \sum_{j=n_2}^{n} \exp(-\lambda_j(\alpha_{\Delta_3} - \beta')) < B''(\beta') \exp(\lambda_n(\alpha_{\Delta_3} - \beta')) \]

where \( B''(\beta') \) is the sum of the series \( \sum_{j=0}^{\infty} \exp(-2(\alpha_{\Delta_3} - \beta') \lambda_j) \).
**Case 2.** If \( \alpha_{\Delta_3} < -\beta' < 0 \), then

\[
\sum_{j=n_2}^{n} \exp(-\lambda_j (\alpha_{\Delta_3} - \beta')) = \sum_{j=n_2}^{n} \exp(\lambda_j |\alpha_{\Delta_3} - \beta'|) = \exp(\lambda_n |\alpha_{\Delta_3} - \beta'|) \sum_{j=n_2}^{n} \exp(-\lambda_j (\alpha_{\Delta_3} - \beta'))
\]

Since the \( D \)-sequence \( \lambda_n \) is of the type \((\Lambda)\) there exists a finite number strictly positive \( B'(\beta') \) such that

\[
\sum_{n \in \mathbb{N} - \{0\}} \exp(-\lambda_n |\alpha_{\Delta_3} - \beta'|) \leq B'(\beta')
\]

which implies that

\[
\sum_{j=n_2}^{n} \exp(-\lambda_j (\alpha_{\Delta_3} - \beta')) \leq B'(\beta') \exp(\lambda_n |\alpha_{\Delta_3} - \beta'|)
\]

On putting \( B''(\beta') = \text{Max}[B'(\beta'), B''(\beta')] \) we have

\[
\sum_{j=n_2}^{n} \exp(-\lambda_j (\alpha_{\Delta_3} - \beta')) \leq B''(\beta') \exp(\lambda_n |\alpha_{\Delta_3} - \beta'|)
\]  

(2.5)

Using the generalized form of Hadamard three circle theorem \([4]\) we have

\[
\exists \quad \log M_{2,\nu} \leq b \log M_{1,\nu} + (1-b) \log M_{3,\nu} \quad (2.6)
\]

where

\[
M_{1,\nu} = \text{Max} \{|R_{n_{\nu}}(s)| / s \in \text{Fr}(\Delta_i)\}, i = 1, 2, 3
\]

with

\[
R_{n_{\nu}}(s) = f(s) - \sum_{j=1}^{n_{\nu}} P_j(s) \exp(-\lambda_j s)
\]

From (2.4) we have for \( n_{\nu} \geq n_1 \)

\[
M_{1,\nu} \leq B(\beta') \exp\{-\lambda_{n_{\nu} + 1}(\alpha_{\Delta_1} - \beta')\} < B(\beta') \exp\{-(1-b)\lambda_n (\alpha_{\Delta_1} - \beta')\}
\]  

(2.7)

because of (2.3). On putting

\[
B_o = \text{Max} \{|f(s)| / s \in \text{Fr}(\Delta_3)\} + \text{Max} \left\{ \sum_{j=1}^{n_2 - 1} |P_j(s) \exp(-\lambda_j s)| / s \in \text{Fr}(\Delta_3) \right\}
\]

we have from (2.5) for \( n_{\nu} \geq n_2 \),

\[
M_{3,\nu} \leq B_o + B''(\beta') \exp(\lambda_n |\alpha_{\Delta_3} - \beta'|)
\]

Let \( B'(\beta') = \text{Max}(B_o, B''(\beta')) \). Then for \( n_{\nu} \geq n_2 \),
Then using (2.7) and (2.8) in (2.6) we get, for \( n \geq \max\{n_1, n_2\} \):

\[
\log M_{2,n} \leq b \log B(\beta') + (1-b) \log B'_0(\beta') + b(1+\varepsilon_n)(\alpha_{\Lambda_1} - \beta') + (1-b) \alpha_{\Lambda_3} - \beta' \]

Since \( \lim_{n \to \infty} \varepsilon_n = \infty \), we have \( \lim_{n \to \infty} (\alpha_{\Lambda_1} - \beta') = -\infty \), and hence \( \lim_{n \to \infty} \log M_{2,n} = -\infty \) which proves the theorem.

When the polynomial \( P_n(s) \) reduces to a complex number \( a_n, \), we get the famous Ostrowski's theorem [1] for Dirichlet series. Our theorem contains G.L. Luntz' theorem [5] as a particular case when \( P_n(s) = a_n s^{m_n} \).

**COROLLARY.** In theorem 3 if we replace (2.3) by the condition that there exists a sequence \( (\varepsilon_n) \) of strictly positive numbers such that

\[
\lim_{n \to \infty} \varepsilon_n = \infty \quad \text{and} \quad \exists \quad \forall \quad n+1 > (1+\varepsilon_n) \lambda_n, \quad \text{then each point of } (\text{Fr}_{\Delta_o}^*) \cap \mathbb{C} - \mathbb{C}^*
\]

is a singular point for \( f \) defined by (2.2). In particular if \( (\text{Fr}_{\Delta_o}^*) \subset \mathbb{C} - \mathbb{C}^* \), then \( \text{Fr}_{\Delta_o}^* \) is a natural boundary for \( f \).

**PROOF.** Let us suppose that the corollary is false. Then there exists a point \( b \in (\text{Fr}_{\Delta_o}^*) \cap \mathbb{C} - \mathbb{C}^* \) and a disc \( d(b, \rho) \) centred at \( b \) and of radius \( \rho > 0 \) on which \( f \) is holomorphic. As a result of theorem 3 the sequence \( (S_n) \) converges on \( d(b, \rho) \). From remark 2 \( \{f\} \) diverges on \( \mathbb{C} - \mathbb{C}^* - \Delta_o^* \). There exists necessarily points common to \( \mathbb{C} - \mathbb{C}^* - \Delta_o^* \) and \( d(b, \rho) \). For these points there is a contradiction which establishes the corollary.

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