A NOTE ON LOCAL ASYMPTOTIC BEHAVIOR FOR BROWNIAN MOTION IN BANACH SPACES

MOU-HSIUNG CHANG

Department of Mathematics
The University of Alabama in Huntsville
Huntsville, Alabama 35807
U.S.A.

(Received July 26, 1978)

ABSTRACT. In this paper we obtain an integral characterization of a two-sided upper function for Brownian motion in a real separable Banach space. This characterization generalizes that of Jain and Taylor [2] where $B = \mathbb{R}^n$. The integral test obtained involves the index of a mean zero Gaussian measure on the Banach space, which is due to Kuelbs [3]. The special case that when B is itself a real separable Hilbert space is also illustrated.

KEY WORDS AND PHRASES. Gaussian measures on B -spaces, abstract Wiener spaces, covariance operators, Brownian motion in B -space, upper and lower functions, integral test.

AMS(MOS) SUBJECT CLASSIFICATION (1970) CODES. Primary 60J65, 60G17.

1. INTRODUCTION

Let B be a real separable Banach space with norm $||\cdot||$ and let B^* be the topological dual of B . If μ is a mean zero Gaussian measure on B then it is well known from [1] that B contains a Hilbert space H_μ with norm $||\cdot||_\mu$ such that $||\cdot||$ is a measurable norm on H_μ in the sense of [1]. As a consequence, the B norm $||\cdot||$ is weaker than $||\cdot||_\mu$. Thus through a restriction map we have the relation that $B^* \subseteq H^* \equiv H \subseteq B$. Furthermore, it is also shown in [1] that μ is the extension of the canonical normal distribution on H_μ to B and we shall say that μ is generated by H_μ . If K denotes the unit ball of H_μ in the norm $||\cdot||_\mu$, let $\Gamma = \sup_{x \in K} ||x||$. The definition of index of μ , n_1 , is due to Kuelbs [3], where

$$n_1 = \sup \{k: \exists f_1, \dots, f_k \in B^*; f_1, \dots, f_k \text{ orthogonal in } H_\mu;$$

$$||f_j||_{B^*} = 1 \text{ and } ||f_j||_\mu = \Gamma \ (1 \leq j \leq k)\}.$$

It is known from [3] that n_1 exists and is finite and if B itself is a Hilbert space then n_1 on B equals the multiplicity of the maximal eigenvalue of the covariance operator for μ . Let $\{W(t): 0 \leq t < \infty\}$ denote μ -Brownian motion in B . Let ϕ_ε denote the class of functions from $(0, \varepsilon)$ to $[0, \infty)$ such that $\phi(t) \uparrow \infty$ as $t \downarrow 0$ and $t^{1/2} \phi(t) \downarrow 0$ as $t \downarrow 0$.

DEFINITION 1. A function $\phi \in \phi_\varepsilon$ is called an upper function for $\{W(t): t > 0\}$ with respect to the norm $||\cdot||$, if given $t > 0$, there exists a $\delta > 0$ such that $P(||W(t+u) - W(t-u)|| < 2^{1/2}(u+v)^{1/2} \phi(u+v) \Gamma \text{ for all } u, v \geq 0 \text{ with } 0 < u+v < \delta) = 1$. In this case, we say $\phi \in U$. $\phi \in \phi_\varepsilon$ is called a lower function for $\{W(t): t \geq 0\}$ with respect to the norm $||\cdot||$, denoted by $\phi \in L$, if $\phi \notin U$.

In the case that $B = \mathbb{R}^d$, a d -dimensional Euclidean space, Jain and Taylor [2] have shown that $\phi \in \Phi_c$ is an upper function for d -dimension standard Brownian motion $\{W(t): 0 \leq t < \infty\}$ with respect to the Euclidean norm $\|\cdot\|_2$ if and only if $\int_{0+}^{\infty} \frac{[\phi(t)]^{d+2}}{t} e^{-\phi^2(t)/2} dt < \infty$. This integral test for two-sided growth in $\mathbb{R}^d$ is the same as that for one-sided growth in $\mathbb{R}^{d+2}$. In the case that B is an infinite dimensional real separable Banach space, Kuelbs [3] has shown that $\phi(t)$, a nonnegative, non-decreasing continuous function defined for large values of t , is a one-sided upper function for μ -Brownian motion $\{W(t): 0 \leq t < \infty\}$ with respect to some equivalent norm $\|\cdot\|_1$ on B if and only if $\int_{0+}^{\infty} \frac{[\phi(t)]^{n_1}}{t} e^{-\phi^2(t)/2} dt < \infty$ where the ϕ is called one-sided upper function with respect to $\|\cdot\|_1$ if $P(\|W(t)\|_1 > t^{1/2}\phi(t)) = 1$ for only a bounded set of t 's. Based on the results of Jain-Taylor and Kuelbs, it is very natural to conjecture that $\phi \in \Phi_c$ is in U (Definition 1) for $\{W(t): 0 \leq t < \infty\}$ with respect to some equivalent norm $\|\cdot\|_1$ on B if and only if $\int_{0+}^{\infty} \frac{[\phi(t)]^{n_1+2}}{t} e^{-\phi^2(t)/2} dt < \infty$. The main purpose of this note is to verify this conjecture. Throughout this note c will stand for a positive number whose value may change from line to line. The notation $a(h) \sim b(h)$ means $\lim_{h \rightarrow 0} \frac{a(h)}{b(h)} = 1$.

2. MAIN RESULTS

The following useful estimates have been used repeatedly in [3] and they can be verified by the argument similar to that in d -dimensional Euclidean space $\mathbb{R}^d$ [4, p. 222].

LEMMA 2. Let $\{W(t): 0 \leq t < \infty\}$ be Brownian motion in a real separable Banach space B having norm $\|\cdot\|$. Then for all $\lambda, h > 0$

$$P\left(\sup_{0 \leq t_1 < t_2 \leq h} \|W(t_2) - W(t_1)\| > \lambda h^{1/2}\right) \leq \quad (1)$$

$$2P\left(\sup_{0 \leq t \leq h} \|W(t)\| > \lambda h^{1/2}\right) \leq 4P(\|W(h)\| > \lambda h^{1/2}).$$

We have the following integral test for a two-sided upper function for $\{W(t): 0 \leq t < \infty\}$:

THEOREM 3. Let $\{W(t): 0 \leq t < \infty\}$ be μ -Brownian motion in a real separable Banach space B having norm $||\cdot||$, and assume $\phi \in \Phi_\epsilon$. Let n_1 denote the index of μ . Then there is an equivalent norm $||\cdot||_1$ on B such that $\sup_{x \in K} ||x||_1 = \Gamma$ and $\phi \in U$ with respect to $||\cdot||_1$ if and only if

$$\int_{0+} \frac{[\phi(t)]^{n_1+2}}{t} e^{-\phi^2(t)/2} dt < \infty. \quad (2)$$

PROOF. The construction of the equivalent norm $||\cdot||_1$ is due to [3], which is defined to be

$$||x||_1 = \max \{ \Gamma ||\pi x||_\mu, ||Qx|| \},$$

where

$$\pi(x) = \sum_{j=1}^{n_1} e_j(x) e_j \quad (x \in B)$$

and

$$Q(x) = x - \pi(x), \quad e_j(\cdot) \text{ denotes the linear function } f_j(\cdot)/\Gamma.$$

Consider the sequence $a_m = e^{-m/\log m}$, $m \geq 2$. Then $\frac{a_m}{a_{m+1}} \sim 1 + (\log m)^{-1}$

as $m \rightarrow \infty$. Let $u_{n,i} = \frac{i}{\log n} a_n$; $v_{n,i} = (1 - \frac{i}{\log n}) a_n$, $0 \leq i \leq \log n$.

Note that for each i , $u_{n,i} + v_{n,i} = a_n$. If $u, v \geq 0$ are sufficiently

small we can choose an n sufficiently large such that $a_{n+1} \leq u+v < a_n$.

Assume that the integral (2) diverges. Define

$$E_{n,i} = \{ \omega: ||W(t+v_{n,i}) - W(t-u_{n,i})||_1 > 2^{1/2} a_n^{1/2} \phi(a_n) \Gamma \} \text{ and}$$

$$F_{n,i} = \{ \omega: ||W(t+v_{n,i}) - W(t-u_{n,i})||_\mu > 2^{1/2} a_n^{1/2} \phi(a_n) \}.$$

Since

$\{\pi W(t): 0 \leq t < \infty\}$ is standard n_1 dimensional Brownian motion in $\pi B = \pi H_\mu$, divergence of the integral (2) implies that

$$\sum_{n=2}^{\infty} \sum_{i=0}^{\log n} P(F_{n,i}) = \infty \text{ and infinitely many of } F_{n,i} \text{ occur.}$$

Consequently infinitely many of $E_{n,i}$ occur with probability one. Thus $\phi \in L$.

Now assume that the integral (2) converges. Let us choose a suitable $i \leq \log n$ such that $u_{n,i} \leq u \leq u_{n,i+1}$ and $v \leq v_{n,i-1}$

Then

$$\begin{aligned} & P(|W(t+v) - W(t-u)|_1 > 2^{1/2} (u+v)^{1/2} \phi(u+v) \Gamma) \\ & \leq P(|W(t+v) - W(t-u)|_1 > 2^{1/2} a_{n+1}^{1/2} \phi(a_{n+1}) \Gamma) \\ & \leq P(0 \leq t_1 < t_2 \leq u_{n,i+1} + v_{n,i-1} \mid |\pi(W(t_2) - W(t_1))|_\mu > 2^{1/2} a_{n+1}^{1/2} \phi(a_{n+1})) \\ & + P(0 \leq t_1 < t_2 \leq u_{n,i+1} + v_{n,i-1} \mid |Q(W(t_2) - W(t_1))| > 2^{1/2} a_{n+1}^{1/2} \phi(a_{n+1}) \Gamma). \end{aligned}$$

Since $\{\pi W(t): 0 \leq t < \infty\}$ is standard n_1 dimensional Brownian motion in $\pi B = \pi H_\mu$, by the same argument as those in Theorem 3.1 [2], we conclude that the first term in the right hand side of the above inequality being zero for infinitely many n and i . As for the second term in the above inequality, we have

$$\begin{aligned} & P(0 \leq t_1 < t_2 \leq u_{n,i+1} + v_{n,i-1} \mid |Q(W(t_2) - W(t_1))| > 2^{1/2} a_{n+1}^{1/2} \phi(a_{n+1}) \Gamma) \equiv P(A_{n,i}) \\ & \leq 4P(|QW(1)| > (\frac{2a_{n+1}}{u_{n,i+1} + v_{n,i-1}})^{1/2} \phi(a_{n+1}) \Gamma) \\ & \leq C \exp\{-\varepsilon \frac{2a_{n+1}}{u_{n,i+1} + v_{n,i-1}} \phi^2(a_{n+1}) \Gamma^2\}, \end{aligned}$$

Where

$\varepsilon < 1/2 \sup_B \int_B \Lambda_n^2(x) \mu^Q(dx)$, $\{\Lambda_j\}$ is a sequence in B^* such that

$$\|\Lambda_j\|_{B^*} = 1 \text{ and } \|x\| = \sup_j |\Lambda_j(x)| \text{ for every } x \in B.$$

The last inequality comes from [3, p. 253].

Since $u_{n,i+1} + v_{n,i-1} = (1 + 2/\log n)a_n \sim (1 + 3/\log n)a_{n+1}$, if

we choose δ be such that $\varepsilon = (1 + \delta)/2\Gamma$ we have

$$\begin{aligned} P(A_{n,i}) &\leq C \exp \{-\delta \phi^2(a_{n+1})\} \exp \{-\phi^2(a_{n+1})\} \\ &\leq C \exp \{-\phi^2(a_{n+1})\} \end{aligned}$$

Thus

$$\begin{aligned} \sum_{n=2}^{\infty} \sum_{i=0}^{\log n} P(A_{n,i}) &< c \sum_{n=2}^{\infty} (\log n) \exp \{-\phi^2(a_{n+1})\} \\ &< c \sum_{n=2}^{\infty} [\phi(a_{n+1})]^{n_1} \exp \{-\phi^2(a_{n+1})\} \\ &< \infty, \text{ since the integral (2) converges} \end{aligned}$$

(see Lemma 2.12 of [2]).

From Lemma 2.15 (i) of [2] we conclude that $P(A_{n,i}, i.o.) = 0$.

Thus

$$P(\sup_{t-u_{n,i+1} \leq t_1 < t_2 \leq t+v_{n,i-1}} |W(t_2) - W(t_1)| \leq 2^{1/2} a_{n+1}^{1/2} \phi(a_{n+1})\Gamma) = 1$$

for all i and n sufficiently large. Thus $\phi \in U$.

In case that B is a real separable Hilbert space, then n_1 equals the multiplicity of the maximal eigenvalue of the covariance operator for μ . We have the same result as those of Theorem 3.

THEOREM 4. Let $\{W(t): 0 \leq t < \infty\}$ be μ -Brownian motion in a real separable Hilbert space H with norm $\|\cdot\|$, and suppose $\phi \in \Phi_\varepsilon$. Then ϕ

is in U with respect to the given norm $||\cdot||$ if and only if

$$\int_0^{\infty} \frac{[\phi(t)]^{n_1+2}}{t} e^{-\phi^2(t)/2} dt < \infty, \quad (3)$$

Where n_1 denotes the multiplicity of the maximal eigenvalue of the covariance operator for μ .

PROOF. Let sequence $\{a_n\}$, $\{u_{n,i}\}$ and $\{v_{n,i}\}$ be the same as those in the proof of Theorem 3. If $u, v \geq 0$ are sufficiently small we choose n sufficiently large such that $a_{n+1} \leq u+v < a_n$ and then fix $i \leq \log n$ such that $u_{n,i} \leq u \leq u_{n,i+1}$ and $v \leq v_{n,i-1}$. If the integral (3) diverges, then we proceed as those in Theorem 3 and conclude that $\phi \notin L$. Now if the integral (3) converges define

$$B_{n,i} = \{\omega: \sup_{t-u_{n,i+1} \leq t_1 < t_2 \leq t+v_{n,i}} |W(t_2) - W(t_1)| > 2^{1/2} a_{n+1}^{1/2} \phi(a_{n+1})\Gamma\}.$$

Then

$$\begin{aligned} P(B_{n,i}) &\leq 4P(|W(1)| > (\frac{2a_{n+1}}{u_{n,i+1}+v_{n,i-1}})^{1/2} \phi(a_{n+1})\Gamma) \\ &\leq C[\phi(a_{n+1})]^{n_1-2} \exp \left(-\frac{a_{n+1} \phi^2(a_{n+1}) \Gamma^2}{(u_{n,i+1}+v_{n,i-1})^\lambda} \right) \\ &\leq C[\phi(a_{n+1})]^{n_1-2} \exp \{-\phi^2(a_{n+1})\} \end{aligned}$$

Where λ is the maximal eigenvalue of the covariance of μ and it is known that $\lambda = \Gamma^2$ [3]. The last inequality comes from [3] and the fact that

$$u_{n,i+1}+v_{n,i-1} \sim \frac{a_{n+1}}{1+3(\log n)^{-1}}.$$

Thus

$$\begin{aligned} \sum_{n=2}^{\infty} \sum_{i=0}^{\log n} P(B_{n,i}) &< C \sum_{n=2}^{\infty} \sum_{i=0}^{\log n} [\phi(a_{n+1})]^{n_1-2} \exp \{-\phi^2(a_{n+1})\} \\ &= C \sum_{n=2}^{\infty} [\phi(a_{n+1})]^{n_1} \exp \{-\phi^2(a_{n+1})\} < \infty \text{ since} \end{aligned}$$

the integral (3) converges. From Lemma 2.15 (i) we have $P(B_{n,i}, \text{i.o.}) = 0$.

That is $P(B_{n,i}^c) = 1$ for sufficiently large i and n . Thus $\phi \in U$.

REFERENCES

1. Gross, L. Abstract Wiener Spaces, Proc. Fifth Berkeley Symposium on Math Statistics and Probability, (1965) 31-42.
2. Jain, N. C. and Taylor, S. J. Local Asymptotic Laws for Brownian Motion. The Annals of Probability, Vol. 1, No. 4 (1973) 527-549.
3. Kuelks, J. Sample Path Behavior for Brownian Motion in Banach Spaces. The Annals of Probability, Vol. 3, No. 2 (1975) 247-261.
4. Taylor, S. J. Exact Asymptotic Estimates of Brownian Path Variation. Duke Math. J. 39, (1972) 219-241.