A COVERING THEOREM FOR ODD TYPICALLY-REAL FUNCTIONS

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ABSTRACT. An analytic function $f(z) = z + a_2 z^2 + \dots$ in $|z| < 1$ is typically-real if $\operatorname{Im} f(z) \operatorname{Im} z \geq 0$. The largest domain G in which each odd typically-real function is univalent (one-to-one) and the domain $\bigcap f(G)$ for all odd typically real functions f are obtained.

KEY WORDS AND PHRASES. Typically-real functions, domain of univalence, covering theorems.

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1. INTRODUCTION.

An analytic function $f(z) = z + a_2 z^2 + \dots$ in the unit disk E ($|z| < 1$) is in the class T of typically-real functions if and only if there exists a nondecreasing function γ on $[0, \pi]$ such that $\gamma(\pi) = 1$, $\gamma(0) = 0$, and

$$f(z) = \int_0^\pi \frac{z d\gamma(t)}{1 - 2z \cos t + z^2}, \quad (1.1)$$

[1]. The function γ when normalized on $(0, \pi)$ by $\gamma(t) = (\gamma(t+) + \gamma(t-))/2$ is uniquely determined by f .

The domain of univalence of the class T is known [2] to be

$$G = \{z : |z - i| < \sqrt{2}\} \cap \{z : |z + i| < \sqrt{2}\} . \quad (1.2)$$

Brannon and Kirwan [3] proved that the largest domain contained in $f(G)$ for every function in T is $|w| < 1/4$.

In this paper we obtain the corresponding results for the class T_0 of odd typically-real functions. Recently Goodman [4] determined the largest domain that is contained in $f(E)$ for every $f \in T$. The analog of this result for the class T_0 is an open problem.

2. The domain of univalence of T_0

THEOREM 2.1. The domain of univalence for T_0 is the domain G of (1.2).

PROOF. Since $T_0 \subset T$, each $f \in T_0$ is univalent in G . The theorem is established, therefore, if we can show that there is a function $f \in T_0$ that is not univalent in any domain D that properly contains G . Let $f(z) = z(1 + z^2)/(1 - z^2)^2 = \frac{1}{2}z/(1 - z)^2 + \frac{1}{2}z/(1 + z)^2$. This function is clearly in T_0 since T is a linear class. The function

$$\zeta = \frac{2z}{1 + z^2} \quad (2.1)$$

maps G onto $|\zeta| < 1$. By the change of variables (2.1), the function f has the form

$$\begin{aligned} f(z) &= \frac{1}{2}z/(1 - 2z + z^2) + \frac{1}{2}z/(1 + 2z + z^2) \\ &= \frac{1}{4}\zeta/(1 - \zeta) + \frac{1}{4}\zeta/(1 + \zeta) = \frac{1}{2}\zeta/(1 - \zeta^2) . \end{aligned}$$

Since $\zeta/(1 - \zeta^2)$ is not univalent in any domain that properly contains $|\zeta| < 1$, we conclude that f is not univalent in any domain that properly contains G .

3. A covering theorem for T_0

THEOREM 3.1. The largest domain U contained in $f(G)$ for every $f \in T_0$ is the domain that includes the origin, is bounded in the right half-plane by $w = \rho e^{i\theta}$,

where

$$\rho = \left\{ \begin{array}{ll} (\cos \theta)/2 & , \quad 0 \leq |\theta| \leq \pi/4, \\ 1/(4|\sin \theta|) & , \quad \pi/4 < |\theta| \leq \pi/2, \end{array} \right\}$$

and is symmetric relative to the imaginary axis.

PROOF. By (1.1) we have

$$-f(-z) = \int_0^\pi \frac{z d\gamma(t)}{1 + 2z \cos t + z^2} = \int_0^\pi \frac{z d[1 - \gamma(\pi - \tau)]}{1 - 2z \cos \tau + z^2}.$$

If $f(z) = -f(-z)$, then by the uniqueness of γ we have $\gamma(t) = 1 - \gamma(\pi - t)$ for $t \in [0, \pi]$. In particular, $\gamma(\pi/2) = 1/2$. For $f \in T_0$, therefore,

$$\begin{aligned} f(z) &= \int_0^{\pi/2} \frac{z d\gamma(t)}{1 - 2z \cos t + z^2} + \int_{\pi/2}^\pi \frac{z d\gamma(t)}{1 - 2z \cos t + z^2} \\ &= \int_0^{\pi/2} \frac{z d\gamma(t)}{1 - 2z \cos t + z^2} + \int_{\pi/2}^0 \frac{z d\gamma(\pi - t)}{1 - 2z \cos(\pi - t) + z^2} \\ &= \int_0^{\pi/2} \left[\frac{z}{1 - 2z \cos t + z^2} + \frac{z}{1 + 2z \cos t + z^2} \right] d\gamma(t). \end{aligned}$$

By the change of variables (2.1), we obtain

$$f(z) = \int_0^{\pi/2} \frac{\zeta d\gamma(t)}{1 - \zeta^2 \cos^2 t}, \quad \gamma(\pi/2) = 1/2, \quad \gamma(0) = 0. \quad (3.1)$$

Let $z \in \partial G$. By (2.1) we have that the corresponding ζ is on the unit circle

$|\zeta| = 1$. For fixed $\zeta = e^{i\theta}$, $-\pi < \theta \leq \pi$, the function $2f(z)$ is by (3.1) in the closed convex hull H of the circular arc $w(s) = e^{i\theta} / (1 - se^{2i\theta})$, $s \in [0, 1]$. For each λ , $0 \leq \lambda \leq 1$, the point $\lambda e^{i\theta} + (1 - \lambda)i/\sin \theta$ is on the linear portion of H . Let $D(\lambda)$ denote the square of the distance from such a point to the origin. If $\theta \neq 0, \pi$, we have

$$D(\lambda) = |\lambda e^{i\theta} + \frac{1}{2}(1-\lambda)i/\sin \theta|^2 = 1 + (1-\lambda^2)/(4 \sin^2 \theta) .$$

This function of λ has a minimum at $\lambda = 1 - 2 \sin^2 \theta$ and $0 \leq \lambda \leq 1$ provided $|\sin \theta| \leq \sqrt{2}/2$. Thus, the distance from any point of H to the origin is not less than $[D(1 - \sin^2 \theta)]^{1/2} = |\cos \theta|$ when $0 < |\theta| \leq \pi/4$ or $3\pi/4 \leq |\theta| < \pi$. For other $\theta \in (-\pi, \pi]$ the distance from any point of H to the origin is $\min\{1, 1/(2|\sin \theta|)\} = 1/(2|\sin \theta|)$ for $\pi/4 < |\theta| < 3\pi/4$ and 1 for $\theta = 0$ or π . Since the convex hull H contains for each $z \in \partial G$ the values of $2f(z)$ for all $f \in T_0$ and since every point of H is the value of $2f(z)$ for some $f \in T_0$, we conclude that U is the exact domain covered by all $f \in T_0$ when $z \in G$.

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