A NOTE ON THE SUBCLASS ALGEBRA

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ABSTRACT. Each irreducible character of the subclass algebra is paired up with its irreducible module.

KEY WORDS AND PHRASES. *Finite group, irreducible character, subclass algebra.*

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INTRODUCTION.

Let G be a finite group and let H be a subgroup of G . If $g \in G$, the subclass of G containing g is the set $E_g = \{hgh^{-1} | h \in H\}$ and the subclass sum containing g is $B_g = \sum_{x \in E_g} x$. The algebra over the complex numbers, K , generated by these subclass sums is called the subclass algebra (denoted by S) associated with G and H .

Let $\{M_1, \dots, M_s\}$ be the irreducible KG -modules with M_j affording the irreducible character, χ_j , of G and let $\{N_1 \dots N_t\}$ be the irreducible KH -modules with N_i

affording the irreducible character ϕ_i , of H . Suppose $\{e_i\}_{i=1}^t$ is a set of primitive orthogonal idempotents of KH and $\{f_i\}_{i=1}^t$ is the set of primitive central orthogonal idempotents of KH where the sets are indexed so that $N_i \cong KHe_i$ and $f_i = (\dim N_i)e_i = \frac{\dim \phi_i}{|H|} \sum_{h \in H} \phi_i(h^{-1})h$. We define the non-negative integers $\{c_{ij}\}$ by $\chi_j \Big|_H = \sum_{i=1}^t c_{ij} \phi_i$.

In [2], it was demonstrated that the irreducible S -modules are $\{e_i M_j\}$.

D. Travis [3] has shown that the irreducible characters of S are parameterized by pairs χ_j, ϕ_i ($c_{ij} \neq 0$) and are given by

$$\psi_{ij}(B_g) = \frac{|E_g|}{|H|} \sum_{h \in H} \chi_j(gh) \phi_i(h^{-1}). \quad (1)$$

Independent of Travis's work we show that the irreducible character afforded by $e_i M_j$ is ψ_{ij} .

LEMMA: $\chi_i(sB_g) = |E_g| \chi(sg) \quad \forall s \in S, \quad \forall g \in G$

PROOF: Since $B_g = \frac{|E_g|}{|H|} \sum_{h \in H} hgh^{-1}$, we have $\chi_i(sB_g) = \frac{|E_g|}{|H|} \sum_{h \in H} \chi_i(shgh^{-1})$
 $= \frac{|E_g|}{|H|} \sum_{h \in H} \chi_i(hsgh^{-1})$ since $hs = sh, \quad \forall h \in H$
 $= |E_g| \chi_i(sg).$

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THEOREM: Let ψ_{ij} be the character afforded by the irreducible S -module $e_i M_j$ ($c_{ij} \neq 0$). Then ψ_{ij} is as defined by equation (1).

PROOF: By proposition 2.3 of [2], we have $M_j \Big|_S = \sum_{k=1}^t (\dim N_k) e_k M_j$
 $= \sum_{k=1}^t f_k M_j.$

Therefore, for $s \in S$, $\chi_j(sf_i) = \sum_{k=1}^t (\dim \phi_k) \psi_{kj}(sf_i)$

$$= (\dim \phi_i) \psi_{ij}(sf_i)$$

$$= (\dim \phi_i) \psi_{ij}(s)$$

since the trace of the action of sf_i on $f_i M_j$ is the same as the trace of the action of s on $f_i M_j$ and the trace of the action of sf_i on $f_k M_j$ ($i \neq k$) is 0.

$$\begin{aligned} \text{Thus } \psi_{ij}(B_g) &= \frac{1}{\dim \phi_i} \chi_j(B_g f_i) \\ &= \frac{|E_g|}{\dim \phi_i} \chi_j(g f_i) \text{ by the Lemma} \\ &= \frac{|E_g|}{|H|} \sum_{h \in H} \chi_j(gh) \phi_i(h^{-1}). \end{aligned}$$

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