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REFLECTION AND TRANSMISSION OF SEISMIC WAVES UNDER INITIAL STRESS AT THE EARTH'S CORE-MANTLE BOUNDARY

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ABSTRACT. In the present paper the influence of the initial stress is shown on the reflection and transmission of P waves at the core-mantle boundary. Taking a particular value of the inherent initial stress, the variations of reflection and transmission coefficients with respect to the angle of emergence are represented by graphs. These graphs when compared with those having no initial stress show that the effect of the initial stress is to produce a reflected P and S waves with numerically higher amplitudes but a transmitted P wave with smaller amplitude. A method is also indicated in this paper to calculate the actual value of the initial stress near the core-mantle boundary by measuring the amplitudes of incident and reflected P waves.

KEY WORDS AND PHRASES. *Seismic Waves, Reflection and Transmission of P Waves.*
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1. INTRODUCTION.

The reflection and transmission of seismic waves at the earth's core-mantle boundary have been discussed by Dana [1], Ibrahim [2] and many other investigators. From their discussions we see that the reflected and transmitted waves are dependent on elastic parameters, densities and the angle of incidence. But the mantle and core contain a considerable amount of initial stress which is compressive and supposed to be hydrostatic in nature (Jeffreys, [3]). The present paper shows that this initial stress has also a significant effect on the reflected and transmitted waves at the core-mantle boundary. The paper is constructed on the assumption that the core is liquid and that there is no discontinuity of initial stress at the core-mantle boundary. For simplicity, only P wave incident from the mantle side has been considered. This P wave produces reflected P (PCP), reflected S (PCS) and transmitted P waves, each of which is influenced by the initial stress. Taking a particular value of the inherent initial stress, the numerical values of reflection and transmission coefficients for different angles of emergence have been calculated and the results are given by graphs. The corresponding graphs when the initial stress is not considered are also given for comparison. From these graphs it is found that the initial stress increases the numerical values of the coefficients for the reflected P and S waves but decreases the same for the transmitted P waves.

It is shown at last that from the expression of the ratio of the coefficients for the reflected and incident P waves we may calculate the actual value of the initial stress near the core-mantle boundary.

2. FORMULATION AND SOLUTION OF THE PROBLEM.

Let us assume that $y = 0$ be the boundary of the earth's core (Fig. 1). The mantle and core are supposed to be homogeneous and isotropic elastic media. Let H be the initial compressive hydrostatic stress just outside and inside the core

including the boundary.

The wave equations with initial hydrostatic stress are the same as those without initial stress (Dey, [4]). They are given by

$$\left.\begin{aligned} \nabla^2\phi &= \frac{\rho}{(\lambda + 2\mu)} \frac{\partial^2\phi}{\partial t^2} \\ \nabla^2\psi &= \frac{\rho}{\mu} \frac{\partial^2\psi}{\partial t^2} \end{aligned}\right\} \qquad (2.1ab)$$

where ρ is the density, λ and μ are Lame's constants and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

We shall consider only P wave incident from the mantle side. The solutions of equations (2.1ab) are

$$\left.\begin{aligned} \phi &= A \exp[ik(ct - x + ay)] + A_1 \exp[ik(ct - x - ay)] \\ \psi &= B_1 \exp[ik(ct - x - by)] , \end{aligned}\right\} \qquad (2.2ab)$$

where k is the wave number, a is connected with the angle of emergence e by the relation a = tan e, and

$$a = \sqrt{c^2/\alpha^2 - 1} , \qquad b = \sqrt{c^2/\beta^2 - 1} ,$$

$$\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}} , \quad \text{and} \quad \beta = \sqrt{\mu/\rho} .$$

For the outer core, which is supposed to be a liquid, the transmitted P wave is given by

$$\phi' = A' \exp ik[(ct - x + a' y)] , \qquad (2.3)$$

where $a' = \sqrt{c^2/\alpha'^2 - 1}$, $\alpha' = \sqrt{\lambda' / \rho'}$, λ' is the Lame constant and ρ' is the density just inside the core.

The boundary conditions require that the vertical displacement v and the incremental boundary force Δf_y per unit initial area are continuous across the

surface $y = 0$ and the incremental tangential force Δf_x per unit initial area vanishes at the same surface. These conditions are equivalent to

$$v = v', \ \Delta f_x = 0, \ \Delta f_y = \Delta f_y' \quad \text{at } y = 0. \tag{2.4}$$

The quantities without and with primes refer to the mantle and core sides respectively.

We write the displacements u, v in terms of the functions ϕ, by the relations

$$u = \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial y} \ , \quad v = \frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial x} \ , \tag{2.5}$$

Δf_x and Δf_y are given in Biot [5]

$$\Delta f_x = s_{12} + H\frac{\partial v}{\partial x} \ , \quad \Delta f_y = s_{22} - H\frac{\partial u}{\partial x} \ , \tag{2.6}$$

where s_{ij} are incremental stresses and are expressed by (Biot, [5])

$$s_{12} = 2\mu\, e_{xy} \ , \quad s_{22} = s + 2\mu\, e_{yy} \tag{2.7}$$

when the initial stress is hydrostatic.

Equations (2.5), (2.6) and (2.7) change the boundary conditions to

$$\left.\begin{aligned} &\frac{A_1}{A} + \frac{1}{a}\frac{B_1}{A} + \frac{a'}{a}\frac{A'}{A} = 1 \\ &\frac{A_1}{A} - \frac{\mu(b^2 - 1) - H}{(2\mu + H)\, a}\frac{B_1}{A} = 1 \\ &\frac{A_1}{A} + \frac{(2\mu + H)\, b}{\mu(b^2 - 1)}\frac{B_1}{A} - \frac{\delta\mu(b^2 + 1) - H}{\mu(b^2 - 1) - H}\frac{A'}{A} = -1, \end{aligned}\right\} \tag{2.8}$$

where $\delta = \rho'/\rho$.

Solving equations (2.8) we obtain

$$\frac{A_1}{A} = \frac{M - N}{M + N} \ , \quad \frac{B_1}{A} = \frac{P}{M + N} \ , \quad \frac{A'}{A} = \frac{Q}{M + N} \ , \tag{2.9}$$

where $M = (2\mu + H)^2 \; ba'a + [\delta\mu(b^2 + 1) - H]\, \mu(b^2 + 1)a,$

$N = [\mu\,(b^2 - 1) - H]^2\, a'\,,$

$P = -2(2\mu + H)\,[\mu(b^2 - 1) - H]\, a'a,$

and $Q = 2\mu\,[\mu(b^2 - 1) - H]\,(b^2 + 1)\, a\,.$

Introducing the non-dimensional parameters $p = \alpha/\beta$, $q = \alpha/\alpha'$ and $\zeta = H/2\mu$ to the relations (2.9) we obtain

$$\frac{A_1}{A} = \frac{M_o - N_o}{M_o + N_o}\,, \quad \frac{B_1}{A} = \frac{P_o}{M_o + N_o}\,, \quad \frac{A'}{A} = \frac{Q_o}{M_o + N_o}\,, \tag{2.10}$$

where $M_o = 4(1 + \zeta)^2(p^2\sec^2 e - 1)^{1/2}(q^2\sec^2 e - 1)^{1/2} \tan e$

$+ \delta(q^2\sec^2 e - 2\,\zeta\,\theta\,)\;(p^2/q^2)p^2\,\sec^2 e \tan e$

$N_o = (p^2\,\sec^2 e - 2 - 2\zeta)^2\;(q^2\,\sec^2 e - 1)^{1/2}$

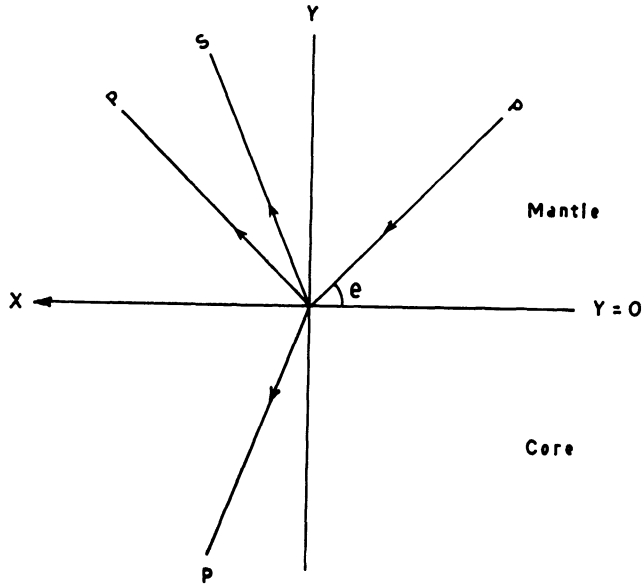
$P_o = -4(1 + \zeta)(p^2\,\sec^2 e - 2 - 2\zeta)(q^2\sec^2 e - 1)^{1/2}\tan e$

and $Q_o = 2p^2\sec^2 e(p^2\sec^2 e - 2 - 2\,\zeta)\,\tan e$ with $\theta = \mu/\lambda'$.

From (2.10) it is clear that A_1/A, B_1/A and A'/A depend on ζ in addition to θ, e and elastic parameters.

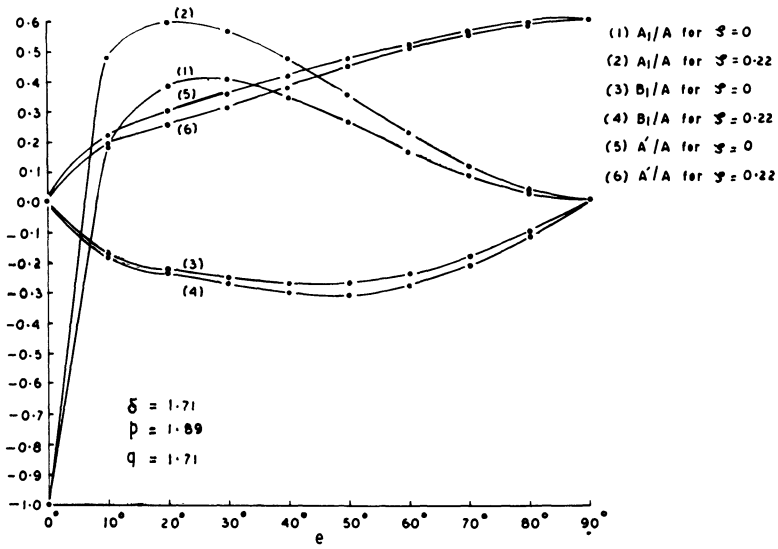
3. NUMERICAL RESULTS AND CONCLUSIONS

Following Bullen [6] we have assumed that $p = 1.89$, $q = 1.71$, $\delta = 1.71$, $\theta = 0.50$ and $\zeta = 0.22$. Taking these values of p, q, δ, θ and ζ , the numerical values of A_1/A, B_1/A and A'/A have been calculated for different values of e between 0° to 90° at the interval of 10°. The results are given graphically in Fig. 2. The corresponding graphs when ζ vanishes are also shown in the same figure. From these graphs we infer the following things:



P WAVE INCIDENT AT THE CORE-MANTLE BOUNDARY

FIG. 1.



VARIATIONS OF REFLECTION AND TRANSMISSION COEFFICIENTS WITH RESPECT TO THE ANGLE OF EMERGENCE

FIG. 2.

At the grazing and normal incidences, i.e. when $e = 0^{\circ}$ and 90° respectively, A_1/A, B_1/A and A'/A are independent of ζ. When $e = 0^{\circ}$, we get $A_1/A = -1$ and $B_1/A = A'/A = 0$. This indicates that there is a total reflection with reversal of phase at the grazing incidence. When $e = 90^{\circ}$, we get $A_1/A = B_1/A = 0$ and $A'/A = 0.58$. This means that no reflection of P wave occurs but a part of it is transmitted through the core-mantle boundary at the normal incidence. When e lies between 0° and 90°, ζ increases the numerical of A_1/A and B_1/A but decreases A'/A. When ζ is not taken into account, A_1/A attains the maximum value 0.4 near $e = 25^{\circ}$. By the consideration of ζ, the maximum value of A_1/A becomes 0.6 near $e = 20^{\circ}$. If ζ is omitted, A_1/A equals A'/A at $e = 34^{\circ}$ approximately. This value of e changes to about 45° by the presence of ζ.

To calculate the actual value of the initial stress, the first equation of (2.10) may be used, which is of the form

$$\frac{A_1}{A} = f(\Delta,\ p,\ q,\ \delta,\ \zeta) \quad , \tag{3.1}$$

where p, q and δ are supposed to be known quantities. The angular distance Δ between the epicentre and station for a surface focus is given by

$$\cos e = \frac{\alpha}{R}\ \frac{dT}{d\Delta} \quad , \tag{3.2}$$

where R is the radius of the core (3470 Km) and $\frac{dT}{d\Delta}$ has been computed from the travel times of PCP (Bullen, [6]).

Hence a careful measurements of A_1/A will lead to the computation of ζ.

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