PEANO COMPACTIFICATIONS AND PROPERTY S METRIC SPACES

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ABSTRACT. Let (X, d) denote a locally connected, connected separable metric space. We say the X is S-metrizable provided there is a topologically equivalent metric ρ on X such that (X, ρ) has Property S, i.e. for any $\epsilon > 0$, X is the union of finitely many connected sets of ρ -diameter less than ϵ . It is well-known that S-metrizable spaces are locally connected and that if ρ is a Property S metric for X , then the usual metric completion $(\tilde{X}, \tilde{\rho})$ of (X, ρ) is a compact, locally connected, connected metric space, i.e. $(\tilde{X}, \tilde{\rho})$ is a Peano compactification of (X, ρ) . There are easily constructed examples of locally connected connected metric spaces which fail to be S-metrizable, however the author does not know of a non-S-metrizable space (X, d) which has a Peano compactification. In this paper we conjecture that: If (P, ρ) a Peano compactification of $(X, \rho|_X)$, X must be S-metrizable. Several (new) necessary and sufficient for a space to be S-metrizable are given, together with an example of non-S-metrizable space which fails to have a Peano compactification.

KEY WORDS AND PHRASES. Property S metrics, Peano spaces, compactifications.

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1. INTRODUCTION.

Throughout this note let (X, d) denote a locally connected, connected separable metric space. We say that X is S-metrizable provided there is a topologically equivalent metric ρ on X such that (X, ρ) has Property S, i.e. for any $\epsilon > 0$, X is the union of finitely many connected sets of ρ -diameter less than ϵ . It is well-known that S-metrizable spaces are locally connected and that if ρ is a Property S metric for X , then the usual metric completion $(\tilde{X}, \tilde{\rho})$ of (X, ρ) is a compact, locally connected, connected metric space, i.e. $(\tilde{X}, \tilde{\rho})$ is a Peano compactification of (X, ρ) [8, p.154].

Property S metric spaces (X, ρ) have been studied extensively in [1, 2, 3, 4, 8]. There are easily constructed examples of locally connected, connected metric spaces which fail to be S-metrizable, however the author does not know of a non-S-metrizable space (X, d) which has a Peano compactification. We therefore ask:

QUESTION 1. If (P, ρ) is a Peano compactification of $(X, \rho|_X)$, must X be S-metrizable?

2. DEFINITIONS AND BASIC RESULTS A space Z is an extension of a space Y if Y is a dense subspace of Z . If Z is an extension of Y , we say that Y is locally connected in Z if Z has a basis consisting of regions (that is, open connected sets) whose intersections with Y are regions in Y . Z is a perfect extension of Y if Z is an extension of Y and whenever a closed subset H of Y separates two sets $A, B \subset Y$ in Y , the set $\text{cl}_Z H$ (the closure of H in Z) separates A, B in Z . [6]

For completeness we include the following:

THEOREM 2.1 [6]. Let Z be an extension of X . Then X is locally connected in Z if and only if Z is a perfect locally connected extension of X .

THEOREM 2.2 [6]. Let (X, d) be a metric space. Then X is S-metrizable if

and only if X has a metrizable compactification Z in which it is locally connected.

THEOREM 2.3 [6]. A topological space is S -metrizable if and only if it has a perfect locally connected metrizable compactification.

THEOREM 2.4 [6]. Let X be a space having a perfect S -metrizable extension. Then X is S -metrizable.

THEOREM 2.5 [5]. Let X be a separable, locally connected, connected rim compact metric space. Then X is S -metrizable.

THEOREM 2.6 [6]. Every countable product of S -metrizable connected spaces $X_1, X_2, \dots$, is S -metrizable.

3. RELATED RESULTS AND QUESTIONS.

THEOREM 3.1. Let (P, d) be a Peano space and let X be a dense, locally connected, connected subset of P . Then there exists a G_δ -subset Y of P containing X such that X is locally connected in Y (as an extension of X).

PROOF. Let n be a positive integer and define $Z_n = \{y \in P: \text{ if } U \text{ is an open connected subset of } P \text{ containing } y \text{ and } \delta(U) < 2^{-n}, \text{ then } U \cap X \text{ is not connected}\}$. (Here $\delta(U)$ denotes the d -diameter of U). We first assert that Z_n is closed. For suppose $y_1, y_2, \dots$, is a sequence in Z_n which converges to $y \in (P \setminus Z_n)$. Since $y \notin Z_n$, there exists an open connected subset U of P containing y and $\delta(U) < 2^{-n}$ and $U \cap Z_n \neq \emptyset$ and this is a contradiction. Hence Z_n is closed.

We next assert $Z_n \cap X = \emptyset$. For let $x \in X$ and let V be an open connected subset of X such that $\delta(\text{cl} V) < 2^{-n}$. Then $U = \text{int } \text{cl} V$ is open in P and contains x and $\delta(U) < 2^{-n}$. Furthermore, $U \cap X$ is connected since $V \subseteq U \cap X \subseteq \text{cl } V$ and V is connected. Thus $x \notin Z_n$ and $Z_n \cap X = \emptyset$.

Clearly $Z_1 \subset Z_2 \subset Z_3 \dots$ is a monotonically increasing sequence and if for each $i \geq 1$, $Y_i = P \setminus Z_i$, $Y = \bigcap_{i=1}^{\infty} Y_i$ is a connected G_δ -subset of P which contains X .

We now assert that X is locally connected in Y , as an extension of X . For let $\epsilon > 0$ and let $y \in Y$. Then there exists a positive integer n so that $\epsilon > 2^{-n}$,

and since $y \notin Z_n$, there exists an open connected subset U of P with $\delta(U) < 2^{-n}$ and such that $U \cap X$ is connected. This implies that $W = \text{int}_Y \text{cl}_Y U$ is an open connected subset of Y . Thus Y has a basis consisting of regions whose intersection with X is connected. This completes the proof.

COROLLARY 3.1.1. Every dense, locally connected, connected G_δ -subset of a Peano continuum is S -metrizable if and only if dense, locally connected, connected subset of a Peano continuum is S -metrizable.

PROOF. This follows from (2.1), (2.4) and (3.1).

Since every nested intersection of countably many sets can be represented as an inverse limit space and since every Y_i above is S -metrizable, by (2.5), we ask:

QUESTION 2. If $\{Y_i, f_{i,j}, \mathbb{N}\}$ is an inverse limit sequence of S -metrizable spaces and continuous maps (bicontinuous injections), must $Y_\infty = \text{inv lim } \{Y_i, f_{i,j}, \mathbb{N}\}$ be S -metrizable?

Of course an affirmative answer to Question 2 would yield an affirmative answer to Question 1.

THEOREM 3.2. Let (X, d) be a locally connected, connected separable metric space, let βX denote the Stone-Čech compactification of X . Then X is S -metrizable if and only if there exists a Peano compactification P of X such that βf , the continuous extension of the identity injection $f: X \rightarrow P$ to βX , is monotone.

PROOF. Recall that a map between compact Hausdorff spaces is monotone if every point inverse is connected. Suppose that (X, d) is S -metrizable, say ρ is an S -metric for X . By (2.3), there exists a Peano compactification P of X and X is locally connected in P . Let $\beta f: \beta X \rightarrow P$ be the continuous extension of the identity map $f: X \rightarrow P$ to βX . We need to show that for $y \in P$, $\beta f^{-1}(y)$ is connected. But since P is a metric space and X is locally connected in P , there exists a neighborhood basis for y in P , $\{U_i\}_{i=1}^\infty$ such that for $i \in \mathbb{N}$, $\text{cl } U_{i+1} \subseteq U_i$ and

$U_i \cap X$ is connected. Then, if $\beta f^{-1}(U_i) = W_i$, $\beta f^{-1}(U_i \cap X) = f^{-1}(U_i \cap X)$ is connected and $W_i \cap X = \beta f^{-1}(U_i \cap X)$. Thus by (1.4) of [7], W_i is connected. It then follows that $\beta f^{-1}(y) = \bigcap_{i=1}^{\infty} \text{cl } W_i$ is connected and that completes the proof of the necessity.

Now suppose (P, ρ) is a Peano compactification of X and $\beta f: \beta X \rightarrow P$ is a monotone map. Let $y \in P$ and let V be an open connected subset of P containing y . Since βf is monotone, $\beta f^{-1}(V) = W$ is a connected open subset of βX . Again, by (1.4) of [7], $W \cap X$ is connected. This implies that $\beta f(W \cap X) = f(W \cap X) = V \cap X$ is connected and so X is locally connected in P . By (2.3), S is S -metrizable.

4. AN EXAMPLE. This is an example which fails to be S -metrizable, however it also fails to have a Peano compactification.

Let L_i be the line in $\mathbb{R}^2$ defined by $L_i = \{(x, y) : y = x/i, 0 \leq x \leq 1\}$ and let $X = \bigcup_{i=1}^{\infty} L_i$ with the relative topology inherited from $\mathbb{R}^2$. We first assert that X is not S -metrizable. For in any (Hausdorff) compactification Z of X , $U_i = L_i \setminus \{(0, 0)\}$ is an open subset of Z and since $A = \{(0, 0)\}$ is compact, A and $B = \bigcup_{i=1}^{\infty} \{(1, i^{-1})\}$ are subsets of X whose closures are disjoint in Z . Thus if Z is a metric space with metric r and the distance from A to $\text{cl}_Z B$ is ϵ , then $\epsilon > 0$. It then follows that no finite collection of connected sets with r -diameter less than $\epsilon/2$ fails to cover Z . Thus r is not a Property S metric for Z and X is not S -metrizable.

We will now show that X fails to have a locally connected metric compactification. Suppose (Z, r) is a locally connected metric compactification of X . Let U and V be open subsets of Z containing $(0, 0)$ such that $\text{cl } U \subseteq V \subseteq (Z \setminus \text{cl } B)$ (B is defined above). Then each L_i intersects $\text{bd } U$ and $\text{bd } V$ and contains a subarc S_i such that $S_i \subseteq (\text{cl } V \setminus U)$ and S_i meets each of $\text{bd } V$ and $\text{bd } U$ in a single point, say $S_i \cap \text{bd } V = \{a_i\}$ and $S_i \cap \text{bd } U = \{b_i\}$. Without loss of generality we may suppose that $\{a_i\}_{i=1}^{\infty}$ converges to a point $a \in \text{bd } V$ and $\{b_i\}_{i=1}^{\infty}$ converges to a point $b \in \text{bd } U$. Then $L = \limsup \{S_i : i \in \mathbb{N}\}$ is a connected set subset of $\text{cl } V \setminus U$ meeting $\text{bd } U$ and $\text{bd } V$ [8, p. 14]. Then since every point of $L \setminus (\text{bd } U \cup \text{bd } V)$ is a limit

point of $\bigcup_{i=1}^{\infty} S_i$ and each S_i is a component of $\text{cl } W \setminus U$, Z fails to be locally connected at any point of $L \setminus (\text{bd } U \cup \text{bd } V)$. Thus X fails to have a Peano compactification.

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