REPRESENTATION OF CERTAIN CLASSES OF DISTRIBUTIVE LATTICES BY SECTIONS OF SHEAVES

U. MADDANA SWAMY and P. MANIKYAMBA

Mathematics Department
Andhra University
Waltair 530 003
INDIA

(Received March 13, 1979 and in revised form July 9, 1979)

ABSTRACT. Epstein and Horn ([6]) proved that a Post algebra is always a P-algebra and in a P-algebra, prime ideals lie in disjoint maximal chains. In this paper it is shown that a P-algebra L is a Post algebra of order $n \geq 2$, if the prime ideals of L lie in disjoint maximal chains each with $n-1$ elements. The main tool used in this paper is that every bounded distributive lattice is isomorphic with the lattice of all global sections of a sheaf of bounded distributive lattices over a Boolean space. Also some properties of P-algebras are characterized in terms of the stalks.

KEY WORDS AND PHRASES. Post Algebra, P-algebra, B-algebra, Heyting Algebra, Stone Lattice, Boolean Space, Sheaf of Distributive Lattices Over a Boolean Space, Prime Ideals Lie in Disjoint Maximal Chains, Regular Chain Base.

1980 Mathematics Subject Classification Codes: Primary 06A35, Secondary 18 F 20.

1. INTRODUCTION.

Epstein ([5]) proved that in a Post algebra of order $n \geq 2$ prime ideals lie in disjoint maximal chains each with $n - 1$ elements. He has also proved that if L is a finite distributive lattice and prime ideals of L lie in disjoint maximal chains each with $n-1$ elements, then L is a Post algebra of order n . Epstein and Horn ([6]) have shown that a Post algebra is always a P-algebra and in a P-algebra prime ideals lie in disjoint maximal chains. It is the main theme of this paper that a P-algebra L is a Post algebra of order $n \geq 2$, if the prime ideals of L lie in disjoint maximal chains each with $n-1$ elements.

The main tool used in this paper is the fact that every bounded distributive lattice is isomorphic with the lattice of all global sections of a sheaf of bounded distributive lattices over a Boolean space ([15] and [9]). It is well known that a P-algebra is always a (double) Heyting algebra, a (double) L-algebra, a pseudocomplemented lattice, a Stone lattice and a normal lattice. We characterize these properties of P-algebras in detail in terms of the stalks of the corresponding sheaf. We give another characterization of Post algebras by regular chain bases.

Throughout this paper, by L we mean a (nontrivial) bounded distributive lattice $(L, \vee, \wedge, 0, 1)$ and $B = B(L)$ the Boolean algebra of complemented elements of L . For any $a \in B$, we denote the complement of a by a' . For any $x, y \in L$, we denote the largest $z \in L$ such that $x \wedge z \leq y$ (if it exists) by $x \rightarrow y$ and the largest element $a \in B$ such that $x \wedge a \leq y$ (if it exists) by $x \Rightarrow y$. If, for every $x, y \in L$, $x \rightarrow y$ ($x \Rightarrow y$) exists, then we say that L is a Heyting algebra (respectively B algebra). Dually, we define $x \leftarrow y$ and $x \Leftarrow y$ respectively. If in a Heyting algebra (B-algebra), $(x \rightarrow y) \vee (y \rightarrow x) = 1$ ($(x \Rightarrow y) \vee (y \Rightarrow x) = 1$) for every $x, y \in L$, then we say that L is an L-algebra

(respectively BL-algebra). For any $x \in L$, if $x \rightarrow 0$ exists, then we say that x has the pseudocomplement and we usually write x^* for $x \rightarrow 0$. If x^* exists for each $x \in L$, then we say that L is pseudocomplemented. The dual of L is denoted by L^d . If both L and L^d are Heyting algebras (B-algebras, L-algebras, BL-algebras), then we say that L is a double Heyting algebra (respectively double B-algebra, double L-algebra, double BL-algebra). L is said to be a P-algebra if L is a BL-algebra. Epstein and Horn proved that L is a P-algebra if and only if L is a double L-algebra ([6], theorem 3.4). For the elementary properties of these types of lattices, we refer to ([2]) and ([6]).

By a sheaf of bounded distributive lattices we mean a triple $(\mathcal{Y}, \pi, X)$ satisfying the following:

- i) $\mathcal{Y}$ and X are topological spaces
- ii) $\pi : \mathcal{Y} \rightarrow X$ is a local homeomorphism
- iii) Each $\pi^{-1}(p)$, $p \in X$ is a bounded distributive lattice;
- iv) the maps $(x, y) \mapsto x \vee y$ and $(x, y) \mapsto x \wedge y$ from $\mathcal{Y} \times \mathcal{Y} = \{(x, y) \in \mathcal{Y} \times \mathcal{Y} / \pi(x) = \pi(y)\}$ into $\mathcal{Y}$ are continuous and
- v) the maps $\hat{0} : p \mapsto 0(p)$ and $\hat{1} : p \mapsto 1(p)$ from $X \rightarrow Q$ are continuous, where $0(p)$ and $1(p)$ are the smallest and largest elements of $\pi^{-1}(p)$ respectively.

We call $\mathcal{Y}$ the sheaf space X the base space and π the projection map. We write $\mathcal{Y}_p$ for $\pi^{-1}(p)$, $p \in X$ and call $\mathcal{Y}_p$ the stalk at p . By a (global) section of the sheaf $(\mathcal{Y}, \pi, X)$ we mean a continuous map $\sigma : X \rightarrow \mathcal{Y}$ such that $\pi \circ \sigma = \text{id}_X$. For any sections σ and τ we write $|\sigma, \tau|$ for the closed set $\{p \in X / \sigma(p) \neq \tau(p)\}$ and we call $|\sigma, 0|$ the support of σ and write $|\sigma|$ for $|\sigma, 0|$. For the preliminary results on sheaf theory, we refer to the pioneering work of Hofmann ([8]).

By $\text{Spec } L$, we mean the (Stone) space Y of all prime ideals of L with the hull-kernel topology; i.e., the topology for which $\{Y_x \mid x \in L\}$ is a base, where for any $x \in L$, $Y_x = \{P \in \text{Spec } L \mid x \notin P\}$. Throughout this paper X denotes $\text{Spec } B$ which is a Boolean space, i.e., a compact, Hausdorff and totally disconnected space. Since $a \mapsto X_a$ is a Boolean isomorphism of B onto the Boolean algebra of all clopen subsets of X , we identify a and X_a and write simply a for X_a . For any $p \in X$, $\mathcal{J}_p$ be the quotient lattice L/θ_p where θ_p is the congruence on L given by

$$(x, y) \in \theta_p \iff x \wedge a = y \wedge a \text{ for some } a \in B - p,$$

and let $\mathcal{J}$ be the disjoint union of all $\mathcal{J}_p$, $p \in X$. For each $x \in L$, define $\hat{x} : X \rightarrow \mathcal{J}$ by $\hat{x}(p) = \theta_p(x)$, the congruence class of θ_p containing x . Topologize $\mathcal{J}$ with the largest topology such that each $\hat{x}$, $x \in L$, is continuous. Define $\pi : \mathcal{J} \rightarrow X$ by $\pi(s) = p$ if $s \in \mathcal{J}_p$. The following theorem is the main tool used in this paper and is due to Subrahmanyam ([15]) (see also [9]).

THEOREM 1.1 $(\mathcal{J}, \pi, X)$ described above is a sheaf of bounded distributive lattices in which each stalk has exactly two complemented elements, viz., $0(p)$ and $1(p)$.

1.2 For any $a \in B$, $p \in X$, $\hat{a}(p) = 1(p)$ if $p \in a$ and $\hat{a}(p) = 0(p)$ if $p \notin a$.

1.3 For any $x, y \in L$ and $a \in B$, $\hat{x}/a = \hat{y}/a$ if and only if $x \wedge a = y \wedge a$.

1.4 $x \mapsto \hat{x}$ is an isomorphism of L onto the lattice $\Gamma(x, \mathcal{J})$ of all global sections of the sheaf $(\mathcal{J}, \pi, X)$. We identify $\hat{x}$ with x and write simply x for $\hat{x}$.

1.5 For any prime ideal P of L , there exists a unique $p \in X$ such that $\{x(p)/x \in P\}$ is a prime ideal of $\mathcal{J}_p$. On the other hand if Q is a prime ideal of $\mathcal{J}_p$, $p \in X$, then $\{x \in L/x(p) \in Q\}$ is a prime ideal of L . This

correspondence exhibits the set of all prime ideals of L as the disjoint union of the sets of prime ideals of the stalks. Moreover, if P and Q are prime ideals of distinct stalks $\mathcal{J}_p$ and $\mathcal{J}_q$, then P and Q are incomparable, when they are regarded as prime ideals of L .

Throughout this paper, by a stalk $\mathcal{J}_p$, $p \in X$, we mean the stalks of the sheaf $(\mathcal{J}, \pi, X)$ described above at p .

2. PSEUDOCOMPLEMENTED LATTICES.

It is well known that a bounded distributive lattice is a Heyting algebra if and only if it is relatively pseudocomplemented; i.e., each interval $[x, y]$, $x \leq y \in L$ is pseudocomplemented ([1]). Also the class of all distributive pseudocomplemented lattices and the class of all Heyting algebras are equationally definable (see [1] and [11]), when we regard the pseudocomplementation and $(x, y) \mapsto (x \rightarrow y)$ as unary and binary operations respectively in L . Thanks to the referee for suggesting a simpler proof of the following.

THEOREM 2.1. L is pseudocomplemented if and only if each stalk $\mathcal{J}_p$, $p \in X$ is pseudocomplemented and the pseudocomplementation $x \mapsto x^*$ is continuous and in this case, the pseudocomplement of $x(p)$ in $\mathcal{J}_p$ is precisely $x^*(p)$ for all $x \in L$.

PROOF. Suppose L is pseudocomplemented. Then it is easily seen that for all x and p , $(x(p))_{\mathcal{J}_p}^*$ exists and is equal to $x^*(p)$. Then it is easy to show that the map $x \mapsto x^*$ is continuous. For the converse, if $x \in L$, the hypothesis implies that the map $f : x \rightarrow \mathcal{J}$ defined by $f(p) = (x(p))^*$ is a global section of $\mathcal{J}$. Therefore, $f = \hat{y}$ for some y and it is clear that $y = x^*$.

If L is a Heyting algebra, then each θ_a , $a \in B$, is compatible with the binary operation $(x, y) \mapsto (x \rightarrow y)$. For, if $a \in B$ and (x, y) and $(x_1, y_1) \in \theta_a$ then $(x \rightarrow x_1) \wedge y \wedge a = (x \rightarrow x_1) \wedge x \wedge a \leq x_1 \wedge a = y_1 \wedge a \leq y_1$, so that $(x \rightarrow x_1) \wedge a \leq (y \rightarrow y_1) \wedge a$. Similarly, we have $(y \rightarrow y_1) \wedge a \leq (x \rightarrow x_1) \wedge a$ and hence

$(x \rightarrow x_1, y \rightarrow y_1) \in \theta_a$. Hence the following theorem and its proof are analogous to the above.

THEOREM 2.2. L is a Heyting algebra if and only if each stalk $\mathcal{J}_p, p \in X$ is a Heyting algebra, and the operation $(s, t) \mapsto (s \rightarrow t)$ of $\mathcal{J}V\mathcal{J}$ into $\mathcal{J}$ is continuous and in this case $x(p) \rightarrow y(p)$ in $\mathcal{J}_p, p \in X$, is equal to $(x \rightarrow y)(p)$ for all $x, y \in L$.

3. NORMAL LATTICES.

DEFINITION 3.1. (Cornish [4]). L is said to be normal if any two distinct minimal prime ideals of L are comaximal and L is said to be relatively normal if each interval $[x, y], x \leq y \in L$ is normal.

For any $x, y \in L$, let $(x, y)_L^*$ be the ideal $\{z \in L / x \wedge z \leq y\}$ of L . For any $x \in L$, we write $(x)_L^*$ for $(x, 0)_L^*$. Cornish ([4]) proved that L is normal if and only if $(x \wedge y)_L^* = (x)_L^* \vee (y)_L^*$ for all $x, y \in L$, and that L is relatively normal if and only if $(x \wedge y, z)_L^* = (x, z)_L^* \vee (y, z)_L^*$ for all $x, y, z \in L$ where $\vee$ stands for the join operation in the lattice of all ideals of L .

THEOREM 3.2. (Speed [13]). A pseudocomplemented distributive lattice is normal if and only if it is a Stone lattice.

THEOREM 3.3. (Balbes and Horn [1]): A Heyting algebra is relatively normal if and only if it is an L -algebra.

THEOREM 3.4. L is normal if and only if each stalk $\mathcal{J}_p, p \in X$, is normal.

PROOF. Suppose L is normal and $p \in X$. Let $u, v \in \mathcal{J}_p$ so that $u = x(p)$ and $v = y(p)$ for some $x, y \in L$. Clearly $(u)_{\mathcal{J}_p}^* \vee (v)_{\mathcal{J}_p}^* \subseteq (u \wedge v)_{\mathcal{J}_p}^*$. Let $t(p) \in (u \wedge v)_{\mathcal{J}_p}^*, t \in L$. Since, $(x \wedge y \wedge t)(p) = x(p) \wedge y(p) \wedge t(p) = 0(p)$ there exists a $a \in B-p$ such that $x \wedge y \wedge t \wedge a = 0$, so that $t \in (x \wedge y \wedge a)_L^* = (x \wedge a)_L^* \vee (y \wedge a)_L^*$ and hence $t = t_1 \vee t_2$ for some $t_1 \in (x \wedge a)_L^*$ and $t_2 \in (y \wedge a)_L^*$. Now $t(p) = t_1(p) \vee t_2(p), t_1(p) \in (u)_{\mathcal{J}_p}^*$ and $t_2(p) \in (v)_{\mathcal{J}_p}^*$. Hence $\mathcal{J}_p$ is normal. Conversely, suppose each stalk $\mathcal{J}_p, p \in X$ is normal. Let $x, y \in L$ and

$z \in (x \wedge y)_L^*$. For each $p \in X$, since $z(p) \in (x(p) \wedge y(p))_{\mathcal{F}_p}^* = (x(p))_{\mathcal{F}_p}^* \vee (y(p))_{\mathcal{F}_p}^*$, there exists $a \in B-p$, t and $s \in L$, such that $a \wedge z = a \wedge (t \vee s)$, $t \wedge x \wedge a = s \wedge y \wedge a = 0$. By the compactness of X , it follows that there exists $a_1, a_2, \dots, a_n \in B$ and $t_1, t_2, \dots, t_n, s_1, \dots, s_n \in L$ such that $\bigvee_{i=1}^n a_i = 1$, $a_i \wedge z = a_i \wedge (t_i \vee s_i)$, $t_i \wedge x \wedge a_i = 0 = s_i \wedge y \wedge a_i$. Now, Put $t = \bigvee_{i=1}^n (t_i \wedge a_i)$ and $s = \bigvee_{i=1}^n (s_i \wedge a_i)$ then, $z = \bigvee_{i=1}^n (z \wedge a_i) = \bigvee_{i=1}^n (a_i \wedge (t_i \vee s_i)) = t \vee s$ and $t \wedge x = \bigvee_{i=1}^n (t_i \wedge a_i \wedge x) = 0 = \bigvee_{i=1}^n (s_i \wedge a_i \wedge y) = s \wedge y$. Hence $(x \wedge y)_L^* \subseteq (x)_L^* \vee (y)_L^*$ and the otherside inclusion is obvious. Hence L is normal.

The proof of the following theorem is analogous to that of the above.

THEOREM 3.5. L is relatively normal if and only if each stalk $\mathcal{F}_p$, $p \in X$, is relatively normal.

DEFINITION 3.6. (Speed [12]). L is said to be a distributive $*$ lattice and denoted by $L \in \Delta^*$ if, for each $x \in L$, there exists $y \in L$ such that $(x)_L^{**} := \{u \in L / u \wedge v = 0 \text{ for every } v \in (x)_L^*\} = (y)_L^*$.

Speed ([12] proved that $L \in \Delta^*$ if and only if, for each $x \in L$, there exists $y \in L$, such that $x \wedge y = 0$ and $x \vee y$ is dense; i.e., $(x \wedge y)_L^* = \{0\}$.

THEOREM 3.7. $L \in \Delta^*$ if and only if (i) $\mathcal{F}_p \in \Delta^*$ for each $p \in X$ and (ii) $\{p \in X / x(p) \text{ is dense in } \mathcal{F}_p\}$ is open for each $x \in L$.

PROOF. Suppose $L \in \Delta^*$ and $x \in L$. There exists $y \in L$ such that $x \wedge y = 0$ and $x \vee y$ is dense in L . Let $p \in X$. Clearly, $x(p) \wedge y(p) = 0(p)$. Also, if $z \in L$, such that $((x(p) \vee y(p)) \wedge z(p) = 0(p)$, then $(x \vee y) \wedge z \wedge a = 0$ for some $a \in B-p$ and hence $z \wedge a = 0$, so that $z(p) = 0(p)$. Hence $x(p) \vee y(p)$ is dense in $\mathcal{F}_p$. Therefore $\mathcal{F}_p \in \Delta^*$. Now, suppose $x(p)$ is dense in $\mathcal{F}_p$. It follows that $y(p) = 0(p)$ and hence there exists $a \in B-p$ such that $y \wedge a = 0$.

We claim that $x(q)$ is dense in $\mathcal{Y}_q$ for all $q \in a$. For, if $q \in a$ and $z(q) \in \mathcal{Y}_q$, $z \in L$, such that $x(q) \wedge z(q) = 0(q)$, then there exists $b \in B - q$ such that $x \wedge z \wedge b = 0$; so that $(x \vee y) \wedge z \wedge a \wedge b = 0$, and hence $z \wedge a \wedge b = 0$ and since $a \wedge b \in B - q$, $z(q) = 0(q)$. Conversely, suppose (i) and (ii) hold and $x \in L$. For each $p \in X$, by (i) and (ii), there exists $y \in L$ and $a \in B - p$ such that $x \wedge y \wedge a = 0$ and $(x \vee y)(q)$ is dense in $\mathcal{Y}_q$ for all $q \in a$. By the usual compactness argument, there exists $y_1, y_2, \dots, y_n \in L$, $a_1, \dots, a_n \in B$ such that $\bigvee_{i=1}^n a_i = 1$, $a_i \wedge a_j = 0$ for $i \neq j$, $x \wedge y_i \wedge a_i = 0$ and $(x \vee y_i)(p)$ is dense in $\mathcal{Y}_p$ for all $p \in a_i$. Now put $y = \bigvee_{i=1}^n (y_i \wedge a_i)$. Then $x \wedge y = 0$ and $x \vee y$ is dense in L . For, $(x \vee y) \wedge z = 0$ for some $z \in L$, then, for all $p \in a_i$, $0(p) = ((x \vee y) \wedge z)(p) = (x(p) \vee y(p)) \wedge z(p)$ and hence $z(p) = 0(p)$ for all $p \in a_i$ and therefore $z = 0$. Hence $L \in \Delta^*$.

4. STONE LATTICES.

For any $p \in X$, since the stalk $\mathcal{Y}_p$ has exactly two complemented elements, $\mathcal{Y}_p$ is a Stone lattice if and only if $\mathcal{Y}_p$ is dense (i.e., if $x(p) \neq 0(p)$, then $(x(p))_{\mathcal{Y}_p}^* = \{0(p)\}$). Hence, by theorem 2.1, 3.2, and 3.4, we have the following.

THEOREM 4.1. Suppose L is pseudocomplemented. Then the following are equivalent.

- (i) L is a Stone lattice
- (ii) L is normal
- (iii) Each stalk $\mathcal{Y}_p$, $p \in X$, is a normal
- (iv) Each stalk $\mathcal{Y}_p$, $p \in X$, is a Stone lattice
- (v) Each stalk $\mathcal{Y}_p$, $p \in X$, is dense.

The following theorem is a consequence of theorem 2.2, 3.3 and 3.5.

THEOREM 4.2. Let L be a Heyting algebra. Then the following are equivalent.

- (i) L is an L -algebra
- (ii) L is relatively normal
- (iii) Each stalk $\mathcal{Y}_p$, $p \in X$, is relatively normal
- (iv) Each stalk $\mathcal{Y}_p$, $p \in X$, is an L -algebra.

Since L is an L -algebra if and only if it is relatively Stone lattice (Theorem 4.11 of [1]) (i.e., each interval is a Stone lattice) in view of theorem 4.1, one may suspect that if L is an L -algebra, then each stalk is relatively dense and hence a chain. This is not true (see 4.4 below), though the converse is proved in the following.

THEOREM 4.3. If L is a Heyting algebra and each stalk is a chain, then L is an L -algebra.

PROOF. If each stalk is a chain, then by theorem 1.5, the prime ideals of L lie in disjoint maximal chains and hence L is relatively normal lattice and hence the theorem follows from theorem 2.3.

EXAMPLE 4.4. Let B_4 be the 4-element Boolean algebra and A be the distributive lattice obtained by adjoining an external element to B_4 as the smallest element. Then A is an L -algebra which is not a chain (Thanks to the referee).

Epstein and Horn ([6]) proved that L is a Stone lattice if and only if L^d is pseudosupplemented and $0 \Leftarrow x \wedge y = (0 \Leftarrow x) \wedge (0 \Leftarrow y)$ for all $x, y \in L$. Now, these two necessary and sufficient conditions for L to be a Stone lattice can be viewed in terms of the stalks as follows.

THEOREM 4.5. L^d is pseudosupplemented if and only if $|x|$ is open for each $x \in L$ and in this case $|x| = 0 \Leftarrow x$ for all $x \in L$.

PROOF. Follows from Lemma 5.2.

For any $p \in X$, let (p) be the smallest ideal of L containing p . The proof of the following theorem is simple.

THEOREM 4.6. For any $p \in X$, the stalk $\mathcal{Y}_p$ is dense if and only if (p) is a prime ideal of L .

It can be easily seen that each stalk $\mathcal{Y}_p$, $p \in X$, is dense if and only if $|x \wedge y| = |x| \cap |y|$ for all $x, y \in L$. Hence from theorem 4.5 and 4.6 and lemma 2.9 of ([7]), we have the following.

THEOREM 4.7. L is a Stone lattice if and only if $|x|$ is open for all $x \in L$ and each stalk $\mathcal{Y}_p$, $p \in X$ is dense.

REMARK 4.8. Swamy and Rama Rao ([10]) proved that a lattice L is a Stone lattice if and only if L is isomorphic to the lattice of all global sections of a sheaf of dense bounded distributive lattices over a Boolean space in which each section has open support (see also [9]). It can be verified, that when L is a Stone lattice, then our sheaf $(\mathcal{Y}, \pi, X)$ coincides with the sheaf constructed in ([10]).

5. P-ALGEBRAS.

The following results interpret B-algebras in sheaf theoretic terminology.

LEMMA 5.1. Let $x, y \in L$. Then $x \Rightarrow y$ exists in B if and only if $\{p \in X / x(p) \leq y(p)\}$ is closed and in this case $x \Rightarrow y = \{p \in X / x(p) \leq y(p)\}$.

PROOF. For any $p \in X$, $x(p) \leq y(p)$ if and only if there exists $a \in B-p$ such that $x \wedge a \leq y$. If $x \Rightarrow y$ exists in B , then, for any $p \in X$, $x(p) \leq y(p)$ if and only if $p \in x \Rightarrow y$. Conversely, if $\{p \in X / x(p) \leq y(p)\}$ is closed, then there exists $a \in B$ such that $p \in a$ if and only if $x(p) \leq y(p)$ for all $p \in X$. Hence $a = x \Rightarrow y$.

The proof of the following is easy.

LEMMA 5.2. For any $x, y \in L$, $|(x, y)|$ is open if and only if there exists a largest element a of B such that $x \wedge a = y \wedge a$.

The following theorem is a consequence of the above lemmas.

THEOREM 5.3. The following are equivalent.

- 1) L is a dual B -algebra
- 2) For any $x, y \in L$, $\{a \in B / x \vee a = y \vee a\}$ is a principal filter of B .
- 3) For any $x, y \in L$, $\{a \in B / x \wedge a = y \wedge a\}$ is a principal ideal of B .
- 4) L is a B -algebra
- 5) $\{p \in X / x(p) \leq y(p)\}$ is closed for every $x, y \in L$.
- 6) $|(x, y)|$ is open for every $x, y \in L$.

THEOREM 5.4. Suppose L is a B -algebra. Then the following are equivalent.

- 1) L is a P -algebra; i.e. L is a BL -algebra
- 2) Each stalk is a chain
- 3) For every $x, y \in L$, there exists $a \in B$ such that $x \wedge a \leq y$ and $y \wedge a' \leq x$.
- 4) For every $x, y \in L$, there exists $a \in B$ such that $x \vee a \geq y$ and $y \vee a' \geq x$.

PROOF. $2 \Leftrightarrow 3$ is proved in ([15]) and $3 \Leftrightarrow 4$ is clear. $1 \Leftrightarrow 2$ follows from lemma 5.1.

6. POST ALGEBRAS.

The following definition is slightly different from that of Chang and Horn ([3]).

DEFINITION 6.1. By a generalized Post algebra, we mean the lattice $C(Z, C)$ of all continuous maps of a Boolean space Z into a discrete bounded chain C where, the operations are pointwise.

THEOREM 6.2. The following are equivalent

- 1) L is a generalized Post algebra.
- 2) There exists a chain C in L such that the natural map $c \mapsto c(p) : C \rightarrow \mathcal{J}_p$ is an isomorphism for all $p \in X$.
- 3) There exists a chain C and, for each $p \in X$, an order isomorphism $\alpha_p : C \rightarrow \mathcal{J}_p$ such that for any $c \in C$ and $x \in L$, $\{p \in X / \alpha_p(c) = x(p)\}$ is open in X .

PROOF. $1 \Rightarrow 2$:

Let $L = C(Z, D)$ where Z is a Boolean space and D is a discrete bounded chain. It is well known that a $\mapsto \chi_a$ is a Boolean isomorphism of the algebra of all clopen subsets of Z onto B , the centre of L , where χ_a is the characteristic function on a . We identify χ_a with a . Also the Stone space X is homeomorphic with Z .

Let C be the set of all constant maps of Z into D . For any $d \in D$, let $\bar{d}$ denote the constant map which maps every element of Z onto d . Then C is a chain in L . Let $p \in X$. Clearly, the natural map $\theta_p: C \rightarrow \mathcal{Y}_p = L/\theta_p$ is a homomorphism.

If $d_1, d_2 \in D$ such that $\bar{d}_1(p) = \bar{d}_2(p)$ then $\bar{d}_1 \wedge a = \bar{d}_2 \wedge a$ for some $a \in B-p$ and hence $d_1 = d_2$. Now, let $x \in L$. Then if $p \in x^{-1}(d)$ for some $d \in D$, since $x: Z \rightarrow D$ is continuous, $x^{-1}(d) \in B-p$ and since $x \wedge x^{-1}(d) = \bar{d} \wedge x^{-1}(d)$, it follows that $(x, \bar{d}) \in \theta_p$. Hence θ_p is an isomorphism.

$2 \Rightarrow 3$: If C is a chain in L and the natural map $\theta_p: C \rightarrow \mathcal{Y}_p$ is an isomorphism for every $p \in X$, then, for any $c \in C$ and $x \in L$. $\{p \in X / \alpha_p(c) = x(p)\} = \{p \in X / c(p) = x(p)\}$ which is open.

$3 \Rightarrow 1$: We first observe that since $\mathcal{Y}_p$ is bounded and α_p is an isomorphism of C onto $\mathcal{Y}_p$, C is also bounded. Let $X = \text{Spec } B$. Define $\theta: L \rightarrow C(X, C)$ by $(\theta(x))(p) = \alpha_p^{-1}(x(p))$ for each $x \in L$ and $p \in X$. Let $c \in C$. Then

$$(\theta(x))^{-1}\{c\} = \{p \in X / \alpha_p^{-1}(x(p)) = c\}$$

$$= \{p \in X / \alpha_p(c) = x(p)\} \text{ is open by (3) and}$$

hence $\theta(x)$ is continuous. Clearly θ is a homomorphism and one-one since α_p^{-1} is so. Now, we will show that θ is onto. Let $f \in C(X, C)$. Define $\sigma: X \rightarrow \mathcal{Y}$ by $\sigma(p) = \alpha_p(f(p))$ for every $p \in X$. We will show that σ is a section. Let $x \in L$ and $a \in B$, then

$$\begin{aligned}\sigma^{-1}(x(a)) &= \{p \in a \mid \alpha_p(f(p)) = x(p)\} \\ &= a \cap \bigcup_{c \in C} \{p \in X \mid f(p) = c\} \cap \{p \in X \mid \alpha_p(c) = x(p)\}.\end{aligned}$$

Since f is continuous, it follows that $\sigma^{-1}(x(a))$ is open. Since $\{x(a) \mid a \in B \text{ and } x \in L\}$ is a base for the topology on $\mathcal{Y}$, it follows that σ is continuous and clearly $\pi \circ \sigma = \text{id}_X$. Therefore, $\sigma = \hat{x}$ for some $x \in L$ and also $\theta(x) = f$. Hence θ is an isomorphism and therefore L is a generalized Post algebra.

THEOREM 6.3. Let $n \geq 2$ and L a P -algebra. Then the following are equivalent.

- 1) L is a Post algebra of order n .
- 2) $\text{Spec } L$ is the disjoint union of maximal chains each with $n-1$ elements.
- 3) Each stalk is a chain with n elements.

PROOF. $1 \Rightarrow 2$ is proved in ([5]).

Since L is a P -algebra, by theorem 4.4, each stalk $\mathcal{Y}_p$, $p \in X$, a chain. Also, by theorem 0.(5), $\text{Spec } L$ is the disjoint union of the chains $\text{Spec } \mathcal{Y}_p$, $p \in X$.

If $\text{Spec } L$ is the disjoint union of all maximal chains C_α , $\alpha \in \Delta$ each with $n-1$ elements, then, for any $p \in X$, $\text{Spec } \mathcal{Y}_p = C_\alpha$ for some $\alpha \in \Delta$. Hence $\text{Spec } \mathcal{Y}_p$ has $n-1$ elements and therefore $\mathcal{Y}_p$ has n elements and hence $(2) \Rightarrow (3)$.

Now, suppose each stalk is a chain with n elements and C_n is the n -element chain $1 < 2 < \dots < n$. For any $p \in X$, let $\mathcal{Y}_p = \{0(p) = x_{1p}(p) < x_{2p}(p) < \dots < x_{np}(p) = 1(p)\}$ where $x_{1p}, x_{2p}, \dots, x_{np} \in L$. Define for any $p \in X$, $\alpha_p: C_n \rightarrow \mathcal{Y}_p$ by $\alpha_p(i) = x_{ip}(p)$ for each $i \in C_n$. Clearly, α_p is an order isomorphism. Let $i \in C_n$, $x \in L$ and $p \in X$ such that $\alpha_p(i) = x(p)$. i.e., $x_{ip}(p) = x(p)$ so that there exists $a \in B-p$ such that $x_{ip}(q) = x(q)$ for all $q \in a$. Since L is a B -algebra and $x_{jp}(p) < x_{kp}(p)$ for all $j < k$, by theorem 5.3, there exists $b \in B-p$ such that $x_{jp}(q) < x_{kp}(q)$ for all $j < k$ and $q \in b$ and hence $x_{ip}(q) = x_{iq}(q)$ for all $i \in C_n$ and $q \in b$. Then $p \in a \wedge b \in B$ and

for any $q \in a \wedge b$, $\alpha_q(i) = x_{iq}(q) = x_{ip}(q) = x(q)$. Hence $\{p \in X / \alpha_p(i) = x(p)\}$ is open for each $i \in C_n$ and $x \in L$.

DEFINITION 6.4. By a chain base C for L we mean a chain C with 0 in L such that L is generated by the centre B and C ; i.e., every $x \in L$ can be

written in the form $\bigvee_{i=1}^n (a_i \wedge c_i)$ for some $a_i \in B$ and $c_i \in C$.

DEFINITION 6.5. A chain base C in L is said to be regular, if, for $c_1 \neq c_2 \in C$ and $a \in B$, $c_1 < c_2$ and $a \wedge c_2 \leq c_1$ imply $a = 0$.

It is proved in ([15]) that a bounded distributive lattice L is a generalized Post algebra if and only if there exists a regular chain base for L . Now, we characterize chain bases and regular chain bases in terms of the stalks. It is also proved in ([15]) that if C is a chain base for L , the natural map $\theta_p: C \rightarrow \mathcal{J}_p$, defined by $\theta_p(c) = c(p)$ is an epimorphism for all $p \in X$. We prove the converse in the following.

THEOREM 6.6. Let C be a chain in L and $0 \in C$. Then $\theta_p: C \rightarrow \mathcal{J}_p$ is an epimorphism for each $p \in X$, if and only if C is a chain base for L .

PROOF. Suppose θ_p is an epimorphism for each $p \in X$ and let $x \in L$. For each $p \in X$, there exists $c_p \in C$ such that $\theta_p(c_p) = x(p)$ i.e., $c_p(p) = x(p)$, so that there exists $a \in B$ such that $c_p \wedge a = x \wedge a$. Therefore, there exists a partition $a_1, a_2, \dots, a_n$ of B and $c_1, c_2, \dots, c_n \in C$ such that $c_i \wedge a_i = x \wedge a_i$ so that $x = x \wedge 1 = x \wedge \bigvee_{i=1}^n a_i = \bigvee_{i=1}^n (x \wedge a_i) = \bigvee_{i=1}^n (c_i \wedge a_i)$. Hence C is a chain base for L .

The following theorem is a straight forward verification.

THEOREM 6.7. Let C be a chain in L . Then the following are equivalent.

- 1) The natural map $\theta_p: C \rightarrow \mathcal{J}_p$ is one for all $p \in X$.
- 2) For any $c_1 \neq c_2 \in C$ and $a \in B$, $c_1 < c_2$ and $a \wedge c_2 \leq c_1$ imply $a = 0$.

- 3) For any $c_1 \neq c_2 \in C$ and $0 \neq a \in B$, $a \wedge c_1 \neq a \wedge c_2$.

By summarizing the above results, we have the following :

THEOREM 6.8. Suppose L is a bounded distributive lattice. Then the following are equivalent.

- 1) L is a generalized Post algebra
- 2) There exists a chain C in L such that the natural map $\emptyset_p: C \rightarrow \mathcal{Y}_p$ is an isomorphism for all $p \in X$.
- 3) There exists a chain C and for each $p \in X$, an order isomorphism $\alpha_p: C \rightarrow \mathcal{Y}_p$ such that for any $c \in C$ and $x \in L$, $\{p \in X / \alpha_p(c) = x(p)\}$ is open in X .
- 4) L has a regular chain base.

REMARK. The equivalence of (1) and (4) is established in ([15]) by using the Boolean extension techniques.

REFERENCES

- [1] Balbes, R. and Horn, A., "Stone Lattices". Duke Math.J. **37** (1970) 537-545.
- [2] Birkhoff, G., "Lattice Theory", Third Edition, AMS. Colloq. Publ. (1967).
- [3] Chang, C.C. and Horn, A., "Prime Ideal Characterization of Generalized Post Algebras", Proc. Symp. in Pure Maths. Vol. 2, (1961) 43-48.
- [4] Cornish, W. H., "Normal Lattices", J. Aust. Math. Soc. **14** (1972), 200-215.
- [5] Epstein, G., "Lattice Theory of Post Algebras". Trans. Amer. Math. Soc. **95** (1960) 300-317.
- [6] Epstein, G. and Horn, A., "P-algebras an Abstraction of Post Algebras", Algebra Universalis, Vol. 4 (1974) 195-206.
- [7] Epstein, G., and Horn, A., "Chain Based Lattices", Pacific J. Maths. Vol. 55, No. 1 (1975).
- [8] Hofmann, K. H., "Representation of Algebras by Continuous Sections". Bull. Amer. Math. Soc. **78** (1972) 291-373.
- [9] Maddana Swamy, U., "Representation of Algebras by Sections of Sheaves". Doctoral Thesis, 1974, Andhra University, Waltair, India.

- [10] Maddana Swamy, U., and Rama Rao, V.V., "Triple and Sheaf Representation of Stone Lattices:", Alg. Univ. Vol. 5 (1975) 104-113.
- [11] Monterio Antonio, "Axiomes Independents Pour Les Algebras De Brouwer". De Revista de la union Matematica Argentina y da la Asociacion. Fisca Argentina, Vol. XVII de Homenaje a BEPPO LEVI, 149-160, Buenos Aires, 1955.
- [12] Speed, T. P., "Some Remarks on a Class of Distributive Lattices". J. Aust. Math. Soc. 9 (1969) 289-296.
- [13] Speed, T. P., "On Stone Lattices", J. Aust. Math. Soc., 9 (1969) 297-307.
- [14] Stone, M. H., "Topological Representations of Distributive Lattices and Brouwerian Logic". Casopis. Pest. Math. Fyz 67 (1937) 1-25.
- [15] Subrahmanyam, N. V., "Lectures on Sheaf Theory". Lecture Notes, Dept. of Mathematics, Andhra University, Waltair, India (1978) (unpublished).