

## K-SPACE FUNCTION SPACES

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**ABSTRACT.** A study is made of the properties on  $X$  which characterize when  $C_{\pi}(X)$  is a  $k$ -space, where  $C_{\pi}(X)$  is the space of real-valued continuous functions on  $X$  having the topology of pointwise convergence. Other properties related to the  $k$ -space property are also considered.

**KEY WORDS AND PHRASES.** Function spaces,  $k$ -spaces, Sequential spaces, Fréchet spaces, Countable tightness,  $k$ -countable,  $\tau$ -countable.

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### 1. INTRODUCTION.

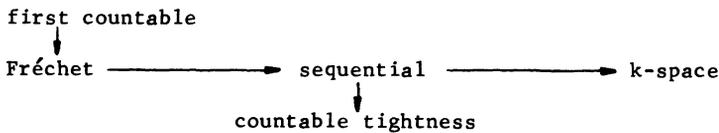
If  $X$  is a topological space, the notation  $C(X)$  is used for the space of all real-valued continuous functions on  $X$ . One of the natural topologies on  $C(X)$  is the topology of pointwise convergence, where subbasic open sets are those of the form

$$[x, V] \equiv \{f \in C(X) \mid f(x) \in V\}$$

for  $x \in X$  and  $V$  open in the space of real numbers,  $\mathbb{R}$ , with the usual topology. The space  $C(X)$  with the topology of pointwise convergence will be denoted by  $C_\pi(X)$ .

For a completely regular space  $X$ ,  $C_\pi(X)$  is first countable, in fact metrizable, if and only if  $X$  is countable [2]. The purpose of this paper is to show to what extent this result can be extended to properties more general than first countability, such as that of being a  $k$ -space. Throughout this paper all spaces will be assumed to be completely regular  $T_1$ -spaces.

We first recall the definitions of certain generalizations of first countability. The space  $X$  is a Fréchet space if whenever  $x \in \bar{A} \subseteq X$ , there exists a sequence in  $A$  which converges to  $x$ . The space  $X$  is a sequential space if the open subsets of  $X$  are precisely those subsets  $U$  such that whenever a sequence converges to an element of  $U$ , the sequence is eventually in  $U$ . Also  $X$  is a  $k$ -space if the closed subsets of  $X$  are precisely those subsets  $A$  such that for every compact subspace  $K \subseteq X$ ,  $A \cap K$  is closed in  $K$ . Finally  $X$  has countable tightness if whenever  $x \in \bar{A} \subseteq X$ , there exists a countable subset  $B \subseteq A$  such that  $x \in \bar{B}$ . The following diagram shows the implications between these properties.



We will show that the Fréchet space, sequential space, and  $k$ -space properties are equivalent for  $C_\pi(X)$ . In order to characterize these properties for  $C_\pi(X)$  in terms of internal properties of  $X$ , we will need to make some additional definitions. Let  $\mathcal{F}(X)$  be the set of all nonempty finite subsets of  $X$ . A collection  $\mathcal{U}$  of open subsets of  $X$  is an open cover for finite subsets of  $X$  if for every  $A \in \mathcal{F}(X)$ , there exists a  $U \in \mathcal{U}$  such that  $A \subseteq U$ . If  $\{u_n\}$  is a sequence of collections of subsets of  $X$ , a string from  $\{u_n\}$  is a sequence  $\{U_n\}$  such that  $U_n \in u_n$

for every  $n \in \mathbb{N}$  ( $\mathbb{N}$  is the set of natural numbers). In addition, we will say that  $\{U_n\}$  is residually covering if for every  $x \in X$ , there exists an  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $x \in U_n$ .

THEOREM 1. The following are equivalent.

- (a)  $C_\pi(X)$  is a Fréchet space.
- (b)  $C_\pi(X)$  is a sequential space.
- (c)  $C_\pi(X)$  is a  $k$ -space.
- (d) Every sequence of open covers for finite subsets of  $X$  has a residually covering string.

PROOF. (d)  $\Rightarrow$  (a). Suppose that every sequence of open covers for finite subsets of  $X$  has a residually covering string. Let  $F$  be a subset of  $C_\pi(X)$ , and let  $f$  be an accumulation point of  $F$  in  $C_\pi(X)$ . Then for every  $n \in \mathbb{N}$  and  $A = \{x_1, \dots, x_k\} \in \mathcal{F}(X)$ , we may choose an

$$f_{n,A} \in F \cap \left[ \left[ x_1, \left( f(x_1) - \frac{1}{n}, f(x_1) + \frac{1}{n} \right) \right] \cap \dots \cap \left[ \left[ x_k, \left( f(x_k) - \frac{1}{n}, f(x_k) + \frac{1}{n} \right) \right] \right].$$

Also define  $U(n,A) = \{x \in X \mid |f_{n,A}(x) - f(x)| < \frac{1}{n}\}$ , which is an open subset of  $X$ .

Then for each  $n \in \mathbb{N}$ , define  $\mathcal{U}_n = \{U(n,A) \mid A \in \mathcal{F}(X)\}$ , which is an open cover for finite subsets of  $X$ . Now  $\{\mathcal{U}_n\}$  has a residually covering string  $\{U(n,A_n)\}$ , so that for every  $n \in \mathbb{N}$ , we may define  $f_n = f_{n,A_n}$ .

We wish to establish that  $\{f_n\}$  converges to  $f$  in  $C_\pi(X)$ . So let  $x \in X$ , and let  $\epsilon > 0$ . There is an  $N \in \mathbb{N}$  with  $N \geq \frac{1}{\epsilon}$  such that for every  $n \geq N$ ,  $x \in U(n,A_n)$ . But then if  $n \geq N$ ,

$$|f_n(x) - f(x)| = |f_{n,A_n}(x) - f(x)| < \frac{1}{n} \leq \frac{1}{N} \leq \epsilon.$$

Therefore  $\{f_n(x)\}$  converges to  $f(x)$  for every  $x \in X$ , so that  $\{f_n\}$  converges to  $f$  in  $C_\pi(X)$ . Hence  $C_\pi(X)$  must be a Fréchet space.

(c)  $\Rightarrow$  (d). Suppose  $X$  has a sequence  $\{\mathcal{U}_n\}$  of open covers for finite subsets such that no string from  $\{\mathcal{U}_n\}$  is residually covering. Let  $V_1 = \mathcal{U}_1$ , and for each  $n > 1$ , let  $V_n$  be an open cover for finite subsets of  $X$  which refines both  $V_{n-1}$

and  $U_n$ . For every  $n \in \mathbb{N}$  and  $A \in \mathcal{F}(X)$ , let  $U(n,A) \in \mathcal{V}_n$  such that  $A \subseteq U(n,A)$ , and let  $f_{n,A} \in C(X)$  be such that  $f_{n,A}(A) = [\frac{1}{n}]$ ,  $f_{n,A}(X \setminus U(n,A)) = \{n\}$ , and  $f_{n,A}(X) \subseteq [\frac{1}{n}, n]$ . Then define

$$F = \{f_{n,A} \mid n \in \mathbb{N} \text{ and } A \in \mathcal{F}(X)\},$$

and also define  $F^* = \overline{F} \setminus \{c_0\}$  in  $C_\pi(X)$ , where  $c_0$  is the constant zero function.

First we establish that  $F^*$  is not closed in  $C_\pi(X)$  by showing that  $c_0$  is an accumulation point of  $F$  in  $C_\pi(X)$ . To do this, let  $W = \prod [x_1, V_1] \cap \dots \cap \prod [x_k, V_k]$  be an arbitrary basic neighborhood of  $c_0$  in  $C_\pi(X)$ . If  $A = \{x_1, \dots, x_k\}$  and  $n \in \mathbb{N}$  such that  $\frac{1}{n} \in V_1 \cap \dots \cap V_k$ , then  $f_{n,A} \in W \cap F$ .

We will then obtain that  $C_\pi(X)$  is not a  $k$ -space, as desired, if we can show that the intersection of  $F^*$  with each compact subspace of  $C_\pi(X)$  is closed in that compact subspace. To this end, let  $K$  be an arbitrary compact subspace of  $C_\pi(X)$ . Then for every  $x \in X$ , the orbit  $\{f(x) \mid f \in K\}$  is bounded in  $\mathbb{R}$ . For every  $x \in X$ , define  $M(x) = \sup \{f(x) \mid f \in K\}$ , and also for every  $m \in \mathbb{N}$ , define  $X_m = \{x \in X \mid M(x) \leq m\}$ . Note that  $X = \bigcup \{X_m \mid m \in \mathbb{N}\}$ , and that for every  $m$ ,  $X_m \subseteq X_{m+1}$ .

Suppose, by way of contradiction, that for every  $m$ ,  $n \in \mathbb{N}$ , there exists a  $k \geq n$  and  $V \in \mathcal{V}_k$  such that  $X_m \subseteq V$ . We define, by induction, a string  $\{U_n\}$  from  $\{U_n\}$ . First there exists a  $k_1 \geq 1$  and  $V_1 \in \mathcal{V}_{k_1}$  such that  $X_1 \subseteq V_1$ . For each  $i = 1, \dots, k_1$ , choose  $U_i \in \mathcal{U}_i$  so that  $V_1 \subseteq U_i$ . Now suppose  $k_m$  and  $U_1, \dots, U_{k_m}$  have been defined. Then there exists a  $k_{m+1} \geq k_m + 1$  and  $V_{m+1} \in \mathcal{V}_{k_{m+1}}$  such that  $X_{m+1} \subseteq V_{m+1}$ . For each  $i = k_m + 1, \dots, k_{m+1}$ , choose  $U_i \in \mathcal{U}_i$  so that  $V_{m+1} \subseteq U_i$ . This defines string  $\{U_n\}$ , which we know to not be residually covering. Let  $x \in X$  be arbitrary. There is an  $m \in \mathbb{N}$  such that  $x \in X_m$ . Let  $n \geq k_m$ . There is a  $j \geq m$  such that  $k_{j-1} + 1 \leq n \leq k_j$ . Then  $x \in X_m \subseteq X_j \subseteq V_j \subseteq U_n$ . But this says that  $\{U_n\}$  is residually covering, which is a contradiction.

We have just established that there exist  $m, n \in \mathbb{N}$  such that for every  $k \geq n$  and for every  $V \in \mathcal{V}_k$ ,  $X_m \not\subseteq V$ . Then define  $M = \max \{m, n\}$ , let  $x_0 \in X$  be

arbitrary, and define  $W = \prod [x_0, (-\frac{1}{M}, \frac{1}{M})]$ , which is a neighborhood of  $c_0$  in  $C_\pi(X)$ . Suppose  $f \in W \cap F$ . Then there exists a  $k \in \mathbb{N}$  and  $A \in \mathcal{F}(X)$  such that  $f = f_{k,A}$ . Since  $\frac{1}{k} \leq f(x_0) < \frac{1}{M}$ , then  $k > M \geq n$ . Thus  $X_m \not\subseteq U(k,A)$ , so that there exists an  $x_1 \in X_m \setminus U(k,A)$ . But then  $f(x_1) = k > M \geq m \geq M(x_1)$ , so that  $f \notin K$ . Therefore  $W \cap F \cap K = \emptyset$ , so that  $c_0$  is not an accumulation point of  $F^* \cap K$  in  $K$ . Hence  $F^* \cap K$  must be closed in  $K$ . Since  $K$  was arbitrary, we obtain that  $C_\pi(X)$  is not a  $k$ -space.  $\square$

**THEOREM 2.**  $C_\pi(X)$  has countable tightness if and only if every open cover for finite subsets of  $X$  has a countable subcover for finite subsets of  $X$ .

**PROOF.** Suppose that every open cover for finite subsets of  $X$  has a countable subcover for finite subsets of  $X$ . Let  $F$  be a subset of  $C_\pi(X)$ , and let  $f$  be an accumulation point of  $F$  in  $C_\pi(X)$ . Then for each  $n \in \mathbb{N}$  and  $A = \{x_1, \dots, x_k\} \in \mathcal{F}(X)$ , choose

$$f_{n,A} \in F \cap \prod [x_1, (f(x_1) - \frac{1}{n}, f(x_1) + \frac{1}{n})] \cap \dots \cap \prod [x_k, (f(x_k) - \frac{1}{n}, f(x_k) + \frac{1}{n})].$$

Also let  $U(n,A) = \{x \in X \mid |f_{n,A}(x) - f(x)| < \frac{1}{n}\}$ , which is an open subset of  $X$ . Then for each  $n \in \mathbb{N}$ ,  $\{U(n,A) \mid A \in \mathcal{F}(X)\}$  is an open cover for finite subsets of  $X$ . So for each  $n \in \mathbb{N}$ , there exists a sequence  $\{A(n,i) \mid i \in \mathbb{N}\}$  from  $\mathcal{F}(X)$  such that  $\{U(n,A(n,i)) \mid i \in \mathbb{N}\}$  is a cover for finite subsets of  $X$ . Then define  $G = \{f_{n,A(n,i)} \mid n, i \in \mathbb{N}\}$ .

To see that  $f \in \overline{G}$ , let  $W = \prod [x_1, V_1] \cap \dots \cap \prod [x_k, V_k]$  be a neighborhood of  $f$  in  $C_\pi(X)$ . Let  $A = \{x_1, \dots, x_k\}$ , and choose  $n \in \mathbb{N}$  so that  $(f(x_j) - \frac{1}{n}, f(x_j) + \frac{1}{n}) \subseteq V_j$  for each  $j = 1, \dots, k$ . Then there is an  $i \in \mathbb{N}$  such that  $A \subseteq U(n,A(n,i))$ . So for each  $x \in A$ ,  $|f_{n,A(n,i)}(x) - f(x)| < \frac{1}{n}$ , and hence  $f_{n,A(n,i)} \in W$ .

Conversely, suppose that  $C_\pi(X)$  has countable tightness, and let  $\mathcal{U}$  be an open cover for finite subsets of  $X$ . For each  $A \in \mathcal{F}(X)$ , let  $U(A) \in \mathcal{U}$  be such that  $A \subseteq U(A)$ . Also for each  $n \in \mathbb{N}$  and  $A \in \mathcal{F}(X)$ , let  $f_{n,A} \in C(X)$  be such that

$f_{n,A}(A) = \{\frac{1}{n}\}$ ,  $f_{n,A}(X \setminus U(A)) = \{n\}$ , and  $f_{n,A}(X) \subseteq [\frac{1}{n}, n]$ . Then define  $F = \{f_{n,A} \mid n \in \mathbb{N} \text{ and } A \in \mathcal{A}(X)\}$ .

Since the constant zero function,  $c_0$ , is an accumulation point of  $F$ , then there is a countable subset  $G$  of  $F$  such that  $c_0 \in \bar{G}$ . There are sequences  $\{n_i\} \subseteq \mathbb{N}$  and  $\{A_i\} \subseteq \mathcal{A}(X)$  so that  $G = \{f_{n_i, A_i} \mid i \in \mathbb{N}\}$ .

To see that  $\{U(A_i) \mid i \in \mathbb{N}\}$  is a cover for finite subsets of  $X$ , let  $A = \{x_i, \dots, x_k\} \in \mathcal{A}(X)$ . Then there exists an  $i \in \mathbb{N}$  such that  $f_{n_i, A_i} \in \llbracket x_i, (-1, 1) \rrbracket \cap \dots \cap \llbracket x_k, (-1, 1) \rrbracket$ . But this means that  $A \subseteq U(A_i)$ , so that  $\{U(A_i) \mid i \in \mathbb{N}\}$  is indeed a cover for finite subsets of  $X$ .  $\square$

Let us now give names to the two properties of  $X$  which are expressed in Theorems 1 and 2. We will call  $X$  k-countable whenever  $C_{\pi}(X)$  is a  $k$ -space, and we will call  $X$   $\tau$ -countable whenever  $C_{\pi}(X)$  has countable tightness. We state some immediate facts about these properties.

PROPOSITION 3. Every countable space is  $k$ -countable.

PROPOSITION 4. Every  $k$ -countable space is  $\tau$ -countable.

PROPOSITION 5. Every  $\tau$ -countable space is Lindelöf.

PROOF. Let  $X$  be  $\tau$ -countable, and let  $\mathcal{U}$  be an open cover of  $X$ . Let  $\mathcal{V}$  be the family of all finite unions of members of  $\mathcal{U}$ . Then  $\mathcal{V}$  is an open cover for finite subsets of  $X$ , so that it has a countable subcover  $\mathcal{W}$  for finite subsets of  $X$ . Each member of  $\mathcal{W}$  is a finite union of members of  $\mathcal{U}$ , so that since  $\mathcal{W}$  covers  $X$ , then  $\mathcal{U}$  has a countable subcover.  $\square$

This means that if  $C_{\pi}(X)$  has countable tightness,  $X$  must be Lindelöf. In particular,  $C_{\pi}(\Omega_0)$  does not have countable tightness, where  $\Omega_0$  is the space of countable ordinals with the order topology. This is in contrast to  $C_{\pi}(\Omega)$ , which we see from the next proposition has countable tightness, where  $\Omega = \Omega_0 \cup \{\omega_1\}$ .

PROPOSITION 6. If  $X^n$  is Lindelöf for every  $n \in \mathbb{N}$ , then  $X$  is  $\tau$ -countable.

PROOF. Let  $X^n$  be Lindelöf for every  $n \in \mathbb{N}$ , and let  $\mathcal{U}$  be an open cover for finite subsets of  $X$ . For each  $n \in \mathbb{N}$ , let  $\mathcal{U}_n = \{U^n \subseteq X^n \mid U \in \mathcal{U}\}$ . Since  $\mathcal{U}$  is an

open cover for finite subsets of  $X$ , then each  $\mathcal{U}_n$  is an open cover of  $X^n$ . So for each  $n \in \mathbb{N}$ ,  $\mathcal{U}$  has a countable subcollection  $\mathcal{V}_n$  such that  $\{U^n \mid U \in \mathcal{V}_n\}$  covers  $X^n$ . But then  $\cup \{\mathcal{V}_n \mid n \in \mathbb{N}\}$  is a countable subcollection of  $\mathcal{U}$  which is a cover for finite subsets of  $X$ .  $\square$

**COROLLARY 7.** Every compact space is  $\tau$ -countable, and every separable metric space is  $\tau$ -countable.

We now examine some properties of  $k$ -countable spaces.

**PROPOSITION 8.** Every closed subspace of a  $k$ -countable space is  $k$ -countable.

**PROOF.** Let  $X$  be a  $k$ -countable space, and let  $Y$  be a closed subspace of  $X$ . Let  $\{\mathcal{V}_n\}$  be a sequence of open covers for finite subsets of  $Y$ . For each  $n \in \mathbb{N}$ , let  $\mathcal{U}_n = \{V \cup (X \setminus Y) \mid V \in \mathcal{V}_n\}$ , which is an open cover for finite subsets of  $X$ . Now  $\{\mathcal{U}_n\}$  has a residually covering string  $\{\mathcal{V}_n \cup (X \setminus Y)\}$ , where each  $V_n \in \mathcal{V}_n$ . But then  $\{\mathcal{V}_n\}$  is a residually covering string from  $\{\mathcal{V}_n\}$ .  $\square$

**PROPOSITION 9.** Every continuous image of a  $k$ -countable space is  $k$ -countable.

**PROOF.** Let  $X$  be  $k$ -countable, and let  $f: X \rightarrow Y$  be a continuous surjection. Let  $\{\mathcal{V}_n\}$  be a sequence of open covers for finite subsets of  $Y$ . For each  $n \in \mathbb{N}$ , let  $\mathcal{U}_n = \{f^{-1}(V) \mid V \in \mathcal{V}_n\}$ , which is an open cover for finite subsets of  $X$ . Now  $\{\mathcal{U}_n\}$  has a residually covering string  $\{f^{-1}(V_n)\}$ , where each  $V_n \in \mathcal{V}_n$ . But then  $\{\mathcal{V}_n\}$  is a residually covering string from  $\{\mathcal{V}_n\}$ .  $\square$

In the next proposition, we use the term covering string, by which we mean a string which is itself a cover of the space.

**PROPOSITION 10.** If  $X$  is  $k$ -countable, then every sequence of open covers of  $X$  has a covering string.

**PROOF.** Let  $\{\mathcal{U}_n\}$  be a sequence of open covers of  $X$ . For each  $n \in \mathbb{N}$ , let

$$\mathcal{V}_n = \{U_n \cup \dots \cup U_{n+k+1} \mid k \in \mathbb{N} \text{ and each } U_i \in \mathcal{U}_i\},$$

which is an open cover for finite subsets of  $X$ . Thus  $\{\mathcal{V}_n\}$  has a residually covering string  $\{\mathcal{V}_n\}$ . Now  $V_1 = U_1 \cup \dots \cup U_{k_1}$  for some  $k_1 \in \mathbb{N}$ . Also  $V_{k_1+1} =$

$U_{k_1+1} \cup \dots \cup U_{k_2}$  for some  $k_2 \in \mathbb{N}$  with  $k_2 > k_1$ . Continuing by induction, we can define an increasing sequence  $\{k_i\}$  such that each  $V_{k_i+1} = U_{k_i+1} \cup \dots \cup U_{k_{i+1}}$ . This defines  $U_n$  for each  $n \in \mathbb{N}$ . To see that  $\{U_n\}$  is a covering string from  $\{U_n\}$  let  $x \in X$ . Then there exists an  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $x \in V_n$ . Since  $\{k_i\}$  is increasing, there is some  $i$  such that  $k_i \geq N$ . Then  $x \in V_{k_i+1} = U_{k_i+1} \cup \dots \cup U_{k_{i+1}}$ , so that  $x$  is indeed in some  $U_n$ .  $\square$

We next give an important example of a space which is not  $k$ -countable.

**EXAMPLE 11.** The closed unit interval,  $I$ , is not  $k$ -countable.

**PROOF.** For each  $n \in \mathbb{N}$ , let  $U_n$  be the set of all open intervals in  $I$  having diameter less than  $\frac{1}{2^n}$ . Suppose  $\{U_n\}$  were to have a covering string  $\{U_n\}$ . Then since  $I$  is connected, there would be a simple chain  $\{U_{n_1}, \dots, U_{n_k}\}$  from 0 to 1. That is,  $0 \in U_{n_1}$ ,  $1 \in U_{n_k}$ , and for each  $1 \leq i \leq k-1$ , there is a  $t_i \in U_{n_i} \cap U_{n_{i+1}}$ . But then

$$\begin{aligned} 1 &\leq |1 - t_{k-1}| + |t_{k-1} - t_{k-2}| + \dots + |t_2 - t_1| + |t_1| \\ &< \frac{1}{2^{n_k}} + \frac{1}{2^{n_{k-1}}} + \dots + \frac{1}{2^{n_2}} + \frac{1}{2^{n_1}} \\ &\leq \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^k} < 1. \end{aligned}$$

This is a contradiction, so that  $\{U_n\}$  cannot have a covering string. Therefore, by Proposition 10,  $I$  is not  $k$ -countable.  $\square$

The next three results are consequences of Example 11.

**EXAMPLE 12.** The Cantor set,  $\mathbb{K}$ , is not  $k$ -countable.

**PROOF.** Since there exists a continuous function from  $\mathbb{K}$  onto  $I$ , then  $\mathbb{K}$  cannot be  $k$ -countable because of Proposition 9 and Example 11.  $\square$

Our next proposition then follows from Example 12 and Proposition 8.

**PROPOSITION 13.** No  $k$ -countable space contains a Cantor set.

**PROPOSITION 14.** Every  $k$ -countable space is 0-dimensional.

PROOF. Let  $X$  be  $k$ -countable, let  $x \in X$ , and let  $U$  be an open neighborhood of  $x$  in  $X$ . Since  $X$  is completely regular, there exists an  $f \in C(X)$  such that  $f(x) = 0$ ,  $f(X \setminus U) = \{1\}$ , and  $f(X) \subseteq I$ . Since  $I$  is not  $k$ -countable by Example 11, and since  $f(X)$  is  $k$ -countable by Proposition 9, then there exists a  $t \in I \setminus f(X)$ . Thus  $[0, t) \cap f(X)$  is both open and closed in  $f(X)$ , so that  $f^{-1}([0, t))$  is an open and closed neighborhood of  $x$  contained in  $U$ .  $\square$

With all these necessary conditions which  $k$ -countable spaces must satisfy, one might wonder whether there exists an uncountable  $k$ -countable space. This is answered by the next two examples.

We will call a space  $X$  virtually countable if there exists a finite subset  $F$  of  $X$  such that for every open subset  $U$  of  $X$  with  $F \subseteq U$ , it is true that  $X \setminus U$  is countable. Notice that a first countable virtually countable space is countable.

PROPOSITION 15. Every virtually countable space is  $k$ -countable.

PROOF. Let  $F$  be a finite subset of  $X$  such that every open  $U$  in  $X$  with  $F \subseteq U$  has countable complement, and let  $\{U_n\}$  be a sequence of open covers for finite subsets of  $X$ . First let  $U_1 \in \mathcal{U}_1$  be such that  $F \subseteq U_1$ . Then  $X \setminus U_1$  is countable; say  $X \setminus U_1 = \{x_{11}, x_{12}, x_{13}, \dots\}$ . Let  $U_2 \in \mathcal{U}_2$  be such that  $F \cup \{x_{11}\} \subseteq U_2$ . Now  $X \setminus U_2$  is also countable; say  $X \setminus U_2 = \{x_{21}, x_{22}, x_{23}, \dots\}$ . Let  $U_3 \in \mathcal{U}_3$  be such that  $F \cup \{x_{11}, x_{12}, x_{21}\} \subseteq U_3$ . Continuing by induction, we may define string  $\{U_n\}$  from  $\{\mathcal{U}_n\}$  such that for each  $n$ ,  $U_n = X \setminus \{x_{n1}, x_{n2}, x_{n3}, \dots\}$  and

$$F \cup \{x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2,n-1}, \dots, x_{n1}\} \subseteq U_{n+1}.$$

To see that every element of  $X$  is residually in  $\{U_n\}$ , let  $x \in X$ . If  $x \in \bigcap_{n=1}^{\infty} U_n$ , then  $x$  is residually in  $\{U_n\}$ . If  $x \notin \bigcap_{n=1}^{\infty} U_n$ , then let  $i$  be the first integer such that  $x \notin U_i$ . Then  $x = x_{ij}$  for some  $j$ , so that for every  $n \geq i + j$ ,  $x \in U_n$ . Therefore  $x$  is residually in  $\{U_n\}$ .  $\square$

EXAMPLE 16. The space of ordinals,  $\Omega$ , which are less than or equal to the first uncountable ordinal is  $k$ -countable.

PROOF. It is easy to see that  $\Omega$  is virtually countable.  $\square$

EXAMPLE 17. The Fortissimo space,  $\mathbb{F}$ , is  $k$ -countable, where  $\mathbb{F}$  is  $\mathbb{R}$  with the following topology: each  $\{t\}$  is open for  $t \neq 0$ , and the open sets containing 0 are the sets containing 0 which have countable complements. Also  $\mathbb{F}^2$  is not Lindelöf, which shows that the converse of Proposition 6 is not true.

PROOF. Obviously  $\mathbb{F}$  is virtually countable. However, an alternate proof can be obtained from known properties of this space. In particular, it follows from [1] that  $C_{\tau}(\mathbb{F})$  is homeomorphic to a  $\Sigma$ -product of copies of  $\mathbb{R}$ , and from [3] that a  $\Sigma$ -product of first countable spaces is a Fréchet space.  $\square$

The spaces in the previous two examples are not first countable. This raises the following question.

QUESTION 18. Is every first countable  $k$ -countable space countable?

One well studied example of an uncountable first countable space which is also a 0-dimensional Lindelöf space and which does not contain a Cantor set is the Sorgenfrey line. However, in our last example we show that this space is not  $k$ -countable, and in fact is not even  $\tau$ -countable.

EXAMPLE 19. The Sorgenfrey line,  $S$ , is not  $\tau$ -countable. This shows that the converse of Proposition 5 is not true.

PROOF. For each  $A \in \mathcal{F}(S)$ , let  $\delta(A) = \frac{1}{2} \min \{|a-a'| \mid a, a' \in A, \text{ with } a \neq a'\}$ , and let  $U(A) = \cup\{[a, a + \delta(A)) \mid a \in A\}$ . Then define  $\mathcal{U} = \{U(\hat{A}) \mid A \in \mathcal{F}(S)\}$ , where  $\hat{A} = A \cup \{-a \mid a \in A\}$ . Clearly  $\mathcal{U}$  is an open cover for finite subsets of  $S$ . Then  $\{U^2 \mid U \in \mathcal{U}\}$  is an open cover of  $S^2$ . But each  $U^2$ , for  $U \in \mathcal{U}$ , intersects the set  $\{(x, y) \in S^2 \mid x + y = 0\}$  on a finite set, so that  $\{U^2 \mid U \in \mathcal{U}\}$  has no countable subcover of  $S^2$ . Therefore no countable subcollection of  $\mathcal{U}$  can cover all doubleton subsets of  $S$ .  $\square$

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