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ON CERTAIN GROUPS OF FUNCTIONS

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ABSTRACT. Let C(X,G) denote the group of continuous functions from a topological space X into a topological group G with the pointwise multiplication and the compact-open topology. We show that there is a natural topology on the collection of normal subgroups $\Delta(X)$ of C(X,G) of the $M_p = \{f \in C(X,G): f(p) = e\}$ which is analogous to the hull-kernel topology on the commutative Banach algegra C(X) of all continuous real or complex-valued functions on X. We also investigate homomorphisms between groups C(X,G) and C(Y,G).

KEY WORDS AND PHRASES. Continuous functions, topological group, compact-open topology, hull-kernel topology, normal subgroups, S-pair, S-topology, Banach algebra, structure space.

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1. INTRODUCTION AND NOTATION.

Suppose X is a compact topological space and suppose C(X) is the algebra of all continuous real or complex-valued functions on X with the usual pointwise operations and the supremum norm. Then C(X) is a regular commutative Banach algebra with identity and X is homeomorphic to the maximal ideal space $\Delta(C(X))$ of the algebra C(X), where $\Delta(C(X))$ is endowed with the Gel'fand topology which coincides with the hull-kernel topology since C(X) is regular, [3]. If X is a topological space and G is a topological group, let C(X,G) be the topological group of all continuous functions from X into G under pointwise multiplication and the compact-open topology. In Section 2 of this paper, we study spaces of normal subgroups of C(X,G). There is a natural topology, analogous to the hull-kernel topology in Banach algebra, for the collection of normal subgroups of the form $M_p = M_p(X,G) = \{f \in C(X,G): f(p) = e\}$, where e is the identity element of G; the resulting topological space will be denoted by $\Delta(X)$. We show that, with some mild restriction on X and G, X is homeomorphic to $\Delta(X)$, and that $\Delta(X^*)$ is the one-point compactification of $\Delta(X)$, where X^* is the one-point compactification of the locally compact space X. Some theorems on homomorphisms and extension of homomorphisms in C(X,G) are considered in Section 3. We also prove a correct version of a theorem originally stated in [7, theorem 8].

All spaces considered in this paper are assumed to be Hausdorff unless specified. For topological spaces X and Y, the function space $F \subset C(X,Y)$ is understood to be endowed with the compact-open topology whenever it is referred to topologically. $I_0(X,G)$, or simply I_0 if no confusion should occur, will denote the identity element of the group C(X,G).

2. THE STRUCTURE SPACES.

For a topological space X and a topological group G, let $\Gamma = C(X,G)$. If X is compact and G is a Lie group, then Γ and M_p, $p \in X$, are in general ℓ_2-manifolds (c.f. [1]). It is easy to see that M_a is locally contractible at I_0, where $a \in X$, if X is a locally compact group locally contractible at a, and that every M_p, $p \in X$, is n-simple for every positive integer n if X is a locally compact contractible space. It is also easy to see that the topological group Γ is a group with equal left and right uniformities if so is the group G, and that, if G is the projective limit of the inverse system of topological groups $\{(G_\alpha, f_{\beta\alpha}): \alpha, \beta \in A\}$, then M_p is the projective limit of the inverse system $\{(M_p(X,G_\alpha), f^p_{\beta\alpha}): \alpha, \beta \in A\}$, where $f^p_{\beta\alpha}(f) = f_{\beta\alpha} \circ f$ for every $f \in M_p(X,G)$.

Throughout this paper the spaces X and G will be subject to the following condition.

DEFINITION 1 [6]. A pair (X,G) of a topological space X and a topological group G is called an S-pair if for each closed subset A of X and $x \notin A$, there exists $f \in \Gamma$ such that $f(x) \neq e$ and $Z(f) = \{x: f(x) = e\} \supset A$.

It is clear that (X,G) is an S-pair if X is completely regular and G is path connected or if X is zero-dimensional. It is also clear that X is completely regular if (X,G) is an S-pair, and that $(\underset{\alpha\in A}{\pi} X_a, \underset{\alpha\in A}{\pi} G_a)$ is also an S-pair whenever (X_α, G_α) is an S-pair for each $\alpha \in A$. Magill called a space X a V-space, [4], if for points p, q, x, and y of X, where $p \neq q$, there exists a continuous function f of X into itself such that $f(p) = x$ and $f(q) = y$, and has shown that every completely regular path connected space and every zero-dimensional space is a V-space. It is easy to see that $(\underset{\alpha\in A}{\pi} X_a, G)$ is an S-pair if each (X_a, G), $a \in A$, is is an S-pair and if G is a V-space. If G is a topological group such that (G,G) is an S-pair, G may not be a V-space. For example, let G_1 be the additive group of real numbers with the usual topology and let G_2 be any non-trivial finite group with the discrete topology, then $(G_1 \times G_2, G_1 \times G_2)$ is an S-pair since (G_1,G_1) and (G_2,G_2) are S-pairs. Since the topological group $G_1 \times G_2$ is not connected with the identity component isomorphic to G_1, $G_1 \times G_2$ is not a V-space as it follows from [4, Theorem 3.5]. It is pointed out in [7] that X is hemicompact and G is metrizable if (X,G) is an S-pair, G is a V-space, and Γ is first countable.

It is well-known (c.f. [2]) that, for every topological space X, there exists a completely regular space Y such that C(Y) is (algebraically) isomorphic to C(X), where C(Z) is the ring of continuous real-valued function on the space Z. Using the similar argument mutatis mutandis as used in the construction of the space Y, it is a straightforward to see that, for every topological space X and a topological group G, there is a completely regular space Y_G such that $C(Y_G,G)$ is continuously isomorphic to C(X,G), and that, in the case G is path connected, (Y_G,G) is an S-pair and the associated space Y_G is independent of the group G within the category of path connected topological groups. The latter means that $Y_{G_1} = Y_{G_2}$ whenever G_1 and G_2 are path connected groups. It follows from the construction of the space Y_G that $X = Y_G$ if (X,G) is an S-pair.

Because of the remarks just made above, we shall now assume that (X,G) is an S-pair.

For a collection $\sum$ of normal subgroups of $\Gamma = C(X,G)$, we define "*" as follows: If $U \subset \sum$ and $U \neq \phi$, let $U^* = \{M \in \sum: M \supset \cap U\}$, let $\phi^* = \phi$.

THEOREM 1. "*" is a closure operator on $\sum$ if and only if whenever $M \in \sum$ and $M \supset M_1 \cap M_2$, where M_1 and M_2 are intersections of some subsets of $\sum$, then either $M \supset M_1$ of $M \supset M_2$.

PROOF: It is clear that $U^* \supset U$, $(U^*)^* = U^*$, $\phi^* = \phi$, and that $U^* \cup V^* \subset (U \cup V)^*$ for subsets U and V of $\sum$. Hence "*" is a closure operator if and only if $U^* \cup V^* \supset (U \cup V)^*$ for subsets U and V of $\sum$. Now if $M_1 = \cap U$, and $M_2 = \cap V$, then $(U \cap V)^* = \{M \in \sum: M \supset M_1 \cap M_2\}$. Hence we have the theorem.

DEFINITION 2. If "*" is a closure operator on $\sum$, we shall refer the resulting topology, not necessarily Hausdorff, on $\sum$ as the S-topology, and the resulting space will be referred to as a G-structure space, or simply structure space, of the space X.

COROLLARY. If $\sum$ admits the S-topology, so is every subset of $\sum$.

REMARK 2. If G is path connected, we may speak of structure spaces for the space X without referring to the group since C(X,G) and C(X,R) are isomorphic in this case.

LEMMA 3. If a collection of normal subgroup $\sum$ of Γ admits the S-topology, then a subset A of $\sum$ is closed if and only if there exists a normal subgroup M_0 of Γ which is the intersection of some subset of $\sum$ such that $A = \{M \in \sum: M \supset M_0\}$. In fact, $M_0 = \cap A$.

PROOF: Suppose $A \subset \sum$ is closed, then $A = \bar{A} = \{M \in \sum: M \supset \cap A = M_0\}$.

Conversely, suppose that there exists a normal subgroup M_0 of Γ, where $M_0 = \cap U$ for some $U \subset \sum$, such that $A = \{M \in \sum: M \supset M_0\}$. Then $\bar{A} = \{M \in \sum: M \supset \cap A\} = A$. Hence A is closed.

THEOREM 4. If a collection of normal subgroups $\sum$ of Γ admits the S-topology, then $\sum$ is Hausdorff if and only if for $M_1, M_2 \in \sum$, $M_1 \neq M_2$, there are I_1 and I_2, where $I_1 = \cap U_1$, $I_2 = \cap U_2$ and $U_1, U_2 \subset \sum$, such that $M_1 \supset I_1$, $M_2 \supset I_2$, $M_1 \not\supset I_2$, $M_2 \not\supset I_1$, and $I_1 \cap I_2 = \cap\sum$.

PROOF: Suppose that $\sum$ is Hausdorff, and let $M_1, M_2 \in \sum$, $M_1 \neq M_2$. Thus there are disjoint open sets U_1 and U_2 in $\sum$ such that $M_1 \subset U_1$, and $M_2 \subset U_2$. If $A_1 = \sum - U_2$, $A_2 = \sum - U_1$, then A_1 and A_2 are closed and $M_1 \in A_1$, $M_2 \in A_2$. Using Lemma 3, we have $A_i = \{M \in \sum: M \supset \cap A_i\}$, $i = 1,2$. If we let $I_i = \cap A_i$, $i = 1,2$, then $M_1 \supset I_1$, $M_2 \supset I_2$, $M_1 \not\supset I_2$, $M_2 \not\supset I_1$ and $I_1 \cap I_2 = \cap\sum$.

Conversely, assume that the stated property holds, and let $M_1, M_2 \in \sum$ such that $M_1 \neq M_2$. Then there are subsets U_1 and U_2 of $\sum$ such that if $I_i = \cap U_i$, $i = 1,2$, $M_1 \supset I_1$, $M_2 \supset I_2$, $M_1 \not\supset I_2$, $M_2 \not\supset I_1$, and $I_1 \cap I_2 = \cap\sum$. Let $B_i = \{M \in \sum: M \supset I_i\}$, $i = 1,2$. Then B_i are closed by Lemma 3, $M_1 \in B_1$, $M_2 \in B_2$, $M_1 \notin B_2$ and $M_2 \notin B_1$. If we let $V_2 = \sum - B_1$, $V_1 = \sum - B_2$, then $M_1 \in V_1$, $M_2 \in V_2$, and $V_1 \cap V_2 = \phi$. To see that $V_1 \cap V_2 = \phi$, it suffices to show that if $M \in \sum$, then either $M \in B_1$ or $M \in B_2$. Now $M \in \sum$ implies $M \supset \cap \sum = I_1 \cap I_2$. This means that either $M \supset I_1$ or $M \supset I_2$ since $\sum$ admits the S-topology. Hence $M \in B_1$ or $M \in B_2$. This completes the proof.

If we denote by $\Delta(X)$ the collection of all normal subgroups of Γ of the form $M_p = \{f \in C(X,G)\colon\ f(p) = e\}$, $p \in X$, then the following theorem states that $\Delta(X)$ admits the S-topology and that the S-topology is Hausdorff if (X,G) is an S-pair.

THEOREM 5. $\Delta(X)$ admits the Hausdorff S-topology.

PROOF: Let U and V be subsets of $\Delta(X)$, and let $O_1 = \{P \in X\colon\ M_p \in U\}$ and $O_2 = \{q \in X\colon\ M_q \in V\}$. It is, by Theorem 1, sufficient to show that, if $M_q \supset (\bigcap_{p \in O_1} M_p) \cap (\bigcap_{k \in O_2} M_k)$, then either $M_q \supset \bigcap_{p \in O_1} M_p$ and $M_q \supset \bigcap_{k \in O_2} M_k$. Suppose otherwise, then there exist $f \in \bigcap_{p \in O_1} M_p - M_q$ and $g \in \bigcap_{k \in O_2} M_k - M_q$. This implies that $q \notin \bar{O}_1$ and $q \notin \bar{O}_2$. For if $q \in \bar{O}_1$, then there is a net $\{q_\alpha\}$ in O_1 such that $q_\alpha \to q$. Then $f(q\ ) \to f(q)$, and hence $f(q) = e$ since $f(q\ ) = e$ for each α. Similarly, $q \notin \bar{O}_2$. Hence $q \notin \overline{O_1 \cup O_2}$. But (X,G) is an S=pair, let $h \in \Gamma$ such that $\overline{O_1 \cup O_2} \subset Z(h)$ but $h(q) \neq e$. This would show that $h \in (\bigcap_{p \in O_1} M_p) \cap (\bigcap_{k \in O_2} M_k)$ but $h \notin M_q$, a contradiction. Hence either $M_q \supset (\bigcap_{p \in O_1} M_p)$ or $M_q \supset (\bigcap_{k \in O_2} M_k)$, and $\Delta(X)$ admits the S-topology.

Next to show that the S-topology is Hausdorff. Let $M_p, M_q \in \Delta(X)$, where $p \neq q$. Since X is T_2, let O_1 and O_2 be open sets in X such that $P \in O_1$, $q \in O_2$ and $O_1 \cap O_2 = \phi$. If $C_2 = X - O_1$ and $C_1 = X - O_2$, then $p \in C_1$ and $q \in C_2$. If $I_1 = \bigcap_{k \in C_1} M_k$ and $I_2 = \bigcap_{k \in O_2} M_k$, then $I_1 \cap I_2 = \cap\Delta(X)$ since $C_1 \cup C_2 = X$, $M_p \supset I_1$, and $M_q \supset I_2$. To see that $M_p \not\supset I_2$ note that $p \notin C_2$, hence there exists $f \in \Gamma$ such that $F(C_2) = e$ but $f(p) \neq e$. Thus $f \in \bigcap_{k \in C_2} M_k$ but $f \notin M_p$. This shows that $M_p \not\supset I_2$. Similarly, we have $M_q \not\supset I_1$. This completes the proof that $\Delta(X)$ is T_2, by Theorem 4.

Note that the S-topology defined above for $\Delta(X)$ is analogous to the hull-kernel topology, which coincides with the Gel'fand topology, on the maximal ideal space of the commutative Banach algebra $C(X)$.

For each $\alpha \in I$, let A_α be a closed set of a structure space $\sum$. Then, by Lemma 3, there exists a normal subgroup M_α of Γ which is the intersection of some subset of $\sum$ such that $A_\alpha = \{M \in \sum: M \supset M_\alpha\}$. If we denoted by $[\bigcup_{\alpha\in I} M_\alpha]$ the normal subgroup of Γ generated by $\bigcup_{\alpha\in I} M_\alpha$, then we have the following lemma whose proof is straightforward and hence omitted.

LEMMA 6. $\bigcap_{\alpha\in I} A_\alpha = \{M \in \sum: M \supset [\bigcup_{\alpha\in I} M]\}$

THEOREM 7. A structure space $\sum$ of X is compact if and only if every collection of normal subgroups $\{N_\alpha\}_{\alpha\in I}$ of Γ, each of which is the intersection of some subset of $\sum$, such that $[\bigcup_{\alpha\in I} N_\alpha] \not\subset M$ for each $M \in \sum$ has a finite subcollection. $\{N_{\alpha_1}, N_{\alpha_2}, \ldots, N_{\alpha_n}\}$ such that $[\bigcup_{i=1}^{n} N_{\alpha_i}] \not\subset M$ for each $M \in \sum$.

PROOF: Suppose $\sum$ is compact, and let $\{N_\alpha\}_{\alpha\in I}$ be a collection of normal subgroups of Γ, each of which is the intersection of some subset of $\sum$, such that $[\bigcup_{\alpha\in I} N_\alpha] \not\subset M$ for each $M \in \sum$. If, for each $\alpha \in I$, let $A_\alpha = \{M \in \sum: M \supset N_\alpha\}$, then A_α is closed in $\sum$, Lemma 3, and $\bigcap_{\alpha\in I} A_\alpha = \{M \in \sum: M \supset [\bigcup_{\alpha\in I} N_\alpha]\} = \phi$. Hence, by the compactness of $\sum$, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $\bigcap_{i=1}^{n} A_{\alpha_i} = \phi$; i.e., there exist $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $\{M \in \sum: M \supset [\bigcup_{i=1}^{n} N_{\alpha_i}]\} = \phi$. Hence $[\bigcup_{i=1}^{n} N_{\alpha_i}] \not\subset M$ for each $M \in \sum$.

Conversely, suppose that $\sum$ has the stated property, and let $\{A_\alpha\}_{\alpha\in I}$ be a collection of closed sets with the finite intersection property, where $A_\alpha = \{M \in \sum: M \supset N_\alpha\}$ and N is the intersection of some subset of $\sum$. Suppose that $\bigcap_{\alpha\in I} A_\alpha = \phi$. Then $\{M \in \sum: M \supset [\bigcup_{\alpha\in I} N_\alpha] = \phi$, hence $[\bigcup_{\alpha\in I} N_\alpha] \not\subset M$ for each $M \in \sum$. Thus, by the hypothesis, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $[\bigcup_{i=1}^{n} N_{\alpha_i}] \not\subset M$ for each $M \in \sum$. This would imply that $\bigcap_{i=1}^{n} A_{\alpha_i} = \phi$, a contradiction. Hence $\sum$ is compact.

COROLLARY. A structure space $\sum$ of X is compact if every normal subgroup N of Γ not contained in any element of $\sum$ contains a finitely generated normal subgroup of N not contained in any element of $\sum$.

PROOF: Assume that the stated property holds in $\sum$, and let $\{N_\alpha\}_{\alpha\in I}$ be a collection of normal subgroups of Γ, each of which is the intersection of some subset of $\sum$, such that $[\bigcup_{\alpha\in I} N_\alpha] \not\subset M$ for every M in $\sum$. Let $N = [\bigcup_{\alpha\in I} N_\alpha]$. Then $N \not\subset M$ for every M in $\sum$, thus N contains a finitely generated normal subgroup B such that $B \not\subset M$ for each $M \in \sum$. Let $B = [a_{\alpha_1}, a_{\alpha_2}, \ldots, a_{\alpha_n}]$, where $a_{\alpha_i} \in N_{\alpha_i}$, $i = 1, 2, \ldots, n$. Then $[\bigcup_{i=1}^{n} N_{\alpha_i}] \not\subset M$ for every M in $\sum$. Hence $\sum$ is compact by Theorem 7.

We shall call a normal subgroup N of Γ free if there is no $p \in X$ such that $f(p) = e$ for each $f \in N$.

COROLLARY. $\Delta(X)$ is compact if every free normal subgroup N of Γ contains a finitely generated free normal subgroup.

THEOREM 8. The mapping ψ: $X \to \Delta(X)$ defined by $\psi(x) = M_x$, $x \in X$, is a homeomorphism.

PROOF: Clearly, ψ is one-to-one and onto.

For the continuity of ψ, let $A \subset \Delta(X)$ be closed. Then there exists a normal subgroup M_0 of Γ such that $A = \{M_x \in \Delta(X)\colon\ M_x \supset M_0\}$. We shall see that $\psi^{-1}(A)$ is closed. For this purpose, let $\{x_\alpha\}$ be a net in $\psi^{-1}(A)$ converging to $x \in X$. Then $M_{x_\alpha} \supset M_0$ for each α. If $M_x \not\supset M_0$, there exists $f \in M_0 - M_x$ which would imply that $f(x) \neq e$, a contradiction since $f(x_\alpha) \to f(x)$ and $f(x) = e$ for each α.

Next to show that ψ is a closed map. Let C be closed in X, and let $M_0 = \bigcap_{x\in C} M_x$. We claim that $\psi(C) = \{M_x \in \Delta(X)\colon\ M_x \supset M_0\}$, which would imply that $\psi(C)$ is closed. It is clear that $\psi(C) \subset \{M_x \in \Delta(X)\colon\ M_x \supset M_0\}$. Now let $M_x \in \{M_x \in \Delta(X)\colon\ M_x \supset M_0\}$. Then $M_x \supset M_0$. Suppose $x \notin C$, then there exists $f \in C(X,G)$ such that $f(C) = e$ but $f(x) \neq e$. Hence $f \in M_0$ but $f \notin M_x$, a contradiction. Thus $x \in C$, and we have $M_x \in \psi(C)$.

If A is a commutative Banach algebra without identity, and if A(e) is the algebra obtained by adjoining an identity to A, then the maximal ideal space

$\Delta(\dot{A}(e))$, with the Gel'fand topology, is the one-point compactification of $\Delta(A)$. Using the previous results, we can also state the following theorem whose proof is now trivial.

THEOREM 9. If X is a locally compact space and X* its one-point compactification, then $\Delta(X^*)$ is the one-point compactification of $\Delta(X)$, and $\Delta(X^*) = \Delta(X) \cup \{M_\infty\}$, where $M_\infty = \{f \in C(X^*,G): f(\infty) = e\}$.

3. HOMOMORPHISMS OF C(X,G).

In this section, we shall study homomorphisms of the group C(X,G) into the group C(Y,G) which leads us to have another version of a theorem originally announced in [7]. We shall also, at the end of the section, consider extensions of homomorphisms of the group C(X,G). All pairs (Z,G) are again assumed to be S-pairs.

DEFINITION 3. (1) A homomorphism ϕ of the group C(Y,G) into the group C(X,G) is said to be a constant-preserving if ϕ maps every constant function on Y into the corresponding constant function on X.

(2) A homomorphism ϕ of the group C(Y,G) into the group C(X,G) which has the property that $\phi^{-1}(\Delta(X)) \subset \Delta(Y)$ is called an F-homomorphism.

It is easy to construct an example of a homomorphism ϕ: $C(Y,G) \to C(X,G)$ which is an F-homomorphism but is not constant-preserving. The following example [5], shows that the converse does not hold either.

EXAMPLE. Let Y = [0,1] be the closed unit interval, and let X = $([-1,1] \times \{0\}) \cup (\{0\} \times (0,1])$ considered as a subspace of R^2. For each $f \in C(Y,R)$, define $\phi(f) \in C(X,R)$ by

$$\phi(f)(t,0) = f(\tfrac{1}{4}(t + 1)), \quad t \in [-1,1]$$

$$\phi(f)(0,s) = f(\tfrac{1}{2}(s + 1)) + f(\tfrac{1}{2}), \quad s \in [0,1]$$

Then ϕ is a constant-preserving isomorphism of C(Y,R) onto C(X,R). If $g \in C(X,R)$, then

$$\phi^{-1}(g)(y) = \begin{cases} g(4y-1,0) & y \in [0,\frac{1}{2}] \\ g(0,2y-1) + g(1,0) - g(0,0), & y \in [\frac{1}{2},1] \end{cases}$$

Now choose $g \in C(X,R)$ such that $Z(g) = \{(0,\frac{1}{2})\}$ and that $g(1,0) - g(0,0) > 0$, then $g \in M_{(0,\frac{1}{2})}$, but $Z(\phi^{-1}(g)) = \phi$. Hence ϕ is not an F-homomorphism.

THEOREM 10. Suppose that ϕ: $C(Y,G) \to C(X,G)$ is a continuous constant-preserving F-homomorphism of $C(Y,G)$ into $C(X,G)$. Then

(1) ϕ induces a one-to-one continuous map of $\Delta(X)$ into $\Delta(Y)$, and

(2) ϕ induces a continuous map j of X into Y such that $j(x) = y$ if and only if $\phi(g)(x) = g(y)$ for each $g \in C(Y,G)$.

PROOF. (1) for each $x \in X$, let hx: $C(X,G) \to G$ be the evaluation map defined by $hx(f) = f(x)$, $f \in C(X,G)$, and let $M_x = \ker hx$. Define $h(x)$: $C(Y,G) \to G$ by $h(x) = hx \circ \phi$, $x \in X$. Then $\ker h(x) = \phi^{-1}(M_x)$, hence $\ker h(x) = M_y$ for some $y \in Y$. Such an y is unique and we have $h(x) = hy$. Now we define a mapping Φ: $(X) \to \Delta(Y)$ by $\Phi(M_x) = M_y$.

Clearly Φ is one-to-one. For the continuity of Φ, let $A =$ $\{M_y \in \Delta(Y)\colon\ M_y \supset M_1\}$, where M_1 is the intersection of some subset U of $\Delta(Y)$, be a closed set in $\Delta(Y)$. We claim that $\Phi^{-1}(A) = \{M_x \in \Delta(X)\colon\ M_x \supset \phi(M_1)\}$ and that $\phi(M_1)$ is the intersection of the subset $\phi(U)$ of $\Delta(X)$. In fact, let $M_x \supset \phi(M_1)$. Then $\phi^{-1}(M_x) \supset M_1$. If $\Phi(M_x) = M_y$, $M_y = \ker hy = \ker (hx \circ \phi) = \phi^{-1}(M_x \supset M_1$, hence $M_y \in A$, thus $M_x = \Phi^{-1}(M_y) \in \Phi^{-1}(A)$. Conversely, let $M_z \in \Phi^{-1}(A)$. Then $\Phi(M_z) \in A$. If $\Phi(M_z) = M_y \in A$ for some $y \in Y$, then $Hy = hz \circ \phi$, hence $\phi(M_1) \subset M_z$. It is easy to see that $\phi(M_1)$ is the intersection of the subset $\phi(U)$ of $\Delta(X)$. Therefore $\Phi^{-1}(A)$ is closed in $\Delta(X)$, and Φ is continuous.

(2) Let the mapping j: $X \to Y$ be defined by $j = \psi_y^{-1} \circ \Phi \circ \psi x$, where

ψ_z: $Z \to \Delta(Z)$ is the mapping of Theorem 8. Then clearly j is continuous and $j(x) = y$ if and only if $\Phi(M_x) = M_y$. To see that $j(x) = y$ if and only if $\phi(g)(x) = g(y)$ for every $g \in C(Y,G)$, let $j(x) = y$. Then $\Phi(M_x) = M_y$. Thus $\ker(hx \circ \phi) = M_y$. Let $g \in C(Y,G)$. If $g \in M_y$, $hx \circ \phi(g) = e$, and we have $\phi(g)(x) = g(y)$. If $g \notin M_y$, there exists $c \in G$ such that $g \in \underline{c}M_y$, where $\underline{c}$ is the constant mapping of X into c, hence $g = \underline{c}k$ for some $k \in My$. Now $hx \circ \phi(g) = hx \circ \phi(\underline{c}k) = hx(\underline{c}\ \phi(k)) = c\phi(k)(x) = c$, while $g(y) = ck(y) = c$. Hence $\phi(g)(x) = g(y)$ for each $g \in C(Y,G)$. Conversely, if $\phi(g)(x) = g(y)$ for each $g \in C(Y,G)$, then, for $g \in C(Y,G)$, $hx \circ \phi(g) = \phi(g)(x) = g(y) = hy(g)$. Thus $\Phi(Mx) = My$, and we have that $j(x) = y$.

REMARK: It is easy to see that, if the mapping ϕ in Theorem 10 is an onto map, then Φ is an embedding.

THEOREM 11. A continuous homomorphism ϕ of $C(Y,G)$ into $C(X,G)$ is a constant-preserving F-homomorphism if and only if there exists $f \in C(X,Y)$ such that $\phi(k) = k \circ f$ for every $k \in C(Y,G)$.

PROOF: It is clear that a homomorphism ϕ of the form $\phi(k) = k \circ f$ for every $k \in C(Y,G)$ is a constant-preserving F-homomorphism. Conversely, if ϕ is a constant-preserving F-homomorphism, and if j is the continuous map of X into Y as defined in Theorem 10, then, for each $k \in C(Y,G)$, $\phi(k)(x) = k(y) = k \circ j(x)$, where $j(x) = y$, Hence $\phi(k) = k \circ j$ for each $k \in C(Y,G)$.

COROLLARY. A homomorphism ϕ of $C(Y,G)$ into $X(X,G)$ is a constant-preserving F-homomorphism if and only if there exists $f \in C(X,Y)$ such that $\phi(k) = k \circ f$ for every $k \in C(Y,G)$.

PROOF: Note that the group topologies for $C(Y,G)$ and $C(X,G)$ are not relevant in the proof of Theorem 10. Hence take discrete topologies for the groups $C(Y,G)$ and $C(X,G)$, then apply the proof of Theorem 11.

As a consequence of the discussions made above, we can now state a correct version of the theorem originally stated in [7, Theorem 8] in the following.

THEOREM 12. If there exists an isomorphism ϕ between groups $C(Y,G)$ and $C(X,G)$ which is constant-preserving such that both ϕ and ϕ^{-1} are F-homomorphisms, then X and Y are homeomorphic.

PROOF: It is clear that ϕ^{-1} is also constant-preserving if ϕ is. Applying the above corollary to ϕ and ϕ^{-1}, there exist functions $j \in C(X,Y)$ and $\ell \in C(Y,X)$ such that $\phi(k) = k \circ j$ for each $k \in C(Y,G)$ and $\phi^{-1}(k) = k \circ \ell$ for each $k \in C(X,G)$. Consequently, we have that $\ell \circ j(x) = x$ and $j \circ \ell(y) = y$ for $x \in X$ and $y \in Y$. To see this suppose that there exists $x \in X$ such that $\ell \circ j(x) \neq x$, then we have $f \in C(X,G)$ such that $f(\ell \circ j(x)) \neq f(x)$ or $f \circ \ell \circ j(x) \neq f(x)$. Hence $(\phi^{-1}(f) \circ j)(x) \neq f(x)$, and thus $\phi(\phi^{-1}(f))(x) \neq f(x)$ which leads to $f(x) \neq f(x)$. Similarly, $j \circ \ell(y) = y$. Hence j is a homeomorphism of X onto Y.

For topological spaces X and Y, it is clear that the space $C(X,Y)$ may be embedded into the space $C(X \times Z,Y)$ as a retract for any space Z, and that every homomorphism of the topological group $C(X,G)$ into a topological group L may be extended to a homomorphism of the topological group $C(X \times Y,G)$ into L for any topological group L. We shall conclude this paper with the following result concerning an extension of F-homomorphisms.

THEOREM 13. Suppose A is a closed subset of X. Then every constant-preserving F-homomorphism h of the topological group $C(G,G)$ into the topological group $C(A,G)$ may be extended to a homomorphism H of the same kind from the topological group $C(G,G)$ into the topological group $C(X,G)$ such that $I \circ H = h$ if every continuous function $f\colon A \to G$ may be continuously extended to all of X, where $I\colon C(X,G) \to C(A,G)$ be the map defined by $I(f) = f \circ i$ for $f \in C(X,G)$, i being the inclusion map of A into X.

PROOF: For necessity, let $f\colon A \quad G$ be any continuous function, and let $f^*\colon C(G,G) \quad C(A,G)$ be the natural homomorphism induced by f, namely $f^*(k) =$

$k \circ f$ for each $k \in C(G,G)$. Then f^* is a constant-preserving F-homomorphism, by Theorem 11. Hence there exists a constant-preserving F-homomorphism H of the topological group $C(G,G)$ into $C(X,G)$ such that $I \circ H = f^*$. Let $\theta \in C(X,G)$ such that $H(k) = k \circ \Theta$ for every $k \in C(G,G)$. If i_d denotes the identity map of G into itself, then, for $a \in A$, $\theta(a) = (i_d \circ \theta)(a) = H(i_d)(a) = H(i_d)(i(a)) = H(i_d) \circ i(a) - I(H(i_d))(a) = (I \circ H)(i_d)(a) = f^*(i_d)(a)\ (i_d \circ f)(a) = f(a)$. Hence θ is an extension of f to all of X.

For sufficiency, assume that every continuous function $f\colon A \to G$ may be extended continuously to all of X, and let $h\colon C(G,G) \to C(A,G)$ be a constant-preserving homomorphism of the topological group $C(G,G)$ into the topological group $C(A,G)$. Then there exists $f \in C(A,G)$ such that $h(k) = k \circ f$ for every $k \in C(G,G)$. If we denote by f the extension of $\hat{f}$ to all of X, define a function H: $C(G,G) \to C(X,G)$ by $H(k) = k \circ \hat{f}$ for each $k \in C(G,G)$. Then H is a constant-preserving F-homomorphism and $I \circ H = h$. This completes the proof.

In particular, if X is a normal space, and A a closed subset of X, then every constant-preserving F-homomorphism h of the topological group $C(R,R)$ into the topological group $C(Z,R)$ may be extended to a homomorphism H of the same kind from the topological group $C(R,R)$ into the topological group $C(X,G)$ such that $I \circ H = h$.

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