ON GENERALIZED QUATERNION ALGEBRAS

GEORGE SZETO

Department of Mathematics
Bradley University
Peoria, Illinois 61625
U.S.A.

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ABSTRACT. Let B be a commutative ring with 1, and $G (= \{\sigma\})$ an automorphism group of B of order 2. The generalized quaternion ring extension $B[j]$ over B is defined by S. Parimala and R. Sridharan such that (1) $B[j]$ is a free B -module with a basis $\{1, j\}$, and (2) $j^2 = -1$ and $jb = \sigma(b)j$ for each b in B . The purpose of this paper is to study the separability of $B[j]$. The separable extension of $B[j]$ over B is characterized in terms of the trace $(= 1 + \sigma)$ of B over the subring of fixed elements under σ . Also, the characterization of a Galois extension of a commutative ring given by Parimala and Sridharan is improved.

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1. INTRODUCTION.

In [6], we studied the separable extension of group rings RG and quaternion rings $R[i,j,k]$ over a ring R with 1. We have shown that $R[i,j,k]$ is a separable extension of R if and only if 2 is a unit in R . Recently, S. Parimala and R. Sridharan ([5]) investigated another class of quaternion ring extensions $B[j]$ over a commutative ring B with 1 and with an automorphism group $G (= \{\sigma\})$ of order 2, where $B[j]$ is a free B -module with a basis $\{1, j\}$, $j^2 = -1$, and $j\sigma(b) = \sigma(b)j$ for each b in B . Their work is based on the following characterization of a Galois extension of a commutative ring ([5], Proposition 1.1): Let A be the set of elements in B fixed under σ . Assume 2 is a unit in A . Then, B is Galois over A if and only if $B_A[j] \cong M_2(B)$, a matrix algebra over B of order 2, where the Galois extension is in the sense of Chase-Harrison-Rosenberg ([2]). The purpose of this paper is to study the separability of $B[j]$. Without the assumption that 2 is a unit in A , we shall characterize the separability of $B[j]$ in terms of the trace $(= 1 + \sigma)$ of B over A . This shows the existence of a separable generalized quaternion ring extension $B[j]$ with 2 not a unit in A . When $\text{Char}(A) = 2$, we shall show that $B[j]$ is a separable extension over B if and only if B is Galois over A . Thus we can improve the above theorem of Parimala and Sridharan. Then, the case in which 2 is a unit will be discussed, and several examples are constructed to illustrate our main results.

2. PRELIMINARIES.

Let us recall some basic definitions as given in [1], [2], [3], [4] and [6]. Let B be a commutative ring containing a subring A with the same identity 1. Then B is called a Galois extension over A ([2], or [3], Chapter 3) with a finite automorphism group G if (1) there exist

elements $\{a_i, b_i \text{ in } B / i = 1, 2, \dots, n \text{ for some integer } n\}$ such that $\sum a_i b_i = 1$ and $\sum a_i \sigma(b_i) = 0$ whenever $\sigma \neq 1$ in G , and (2) $A = \{b \text{ in } B / \sigma(b) = b \text{ for all } \sigma \text{ in } G\}$. The map $\sum \sigma$ is called the trace of B over A denoted by Tr . Let S be a ring (not necessarily commutative) containing a subring R with the same identity 1 . Then S is called a separable extension of R if there exist elements, $\{c_i, d_i \text{ in } S / i = 1, 2, \dots, n \text{ for some integer } n\}$ such that (1) $a(\sum c_i \otimes d_i) = (\sum c_i \otimes d_i)a$ for all a in S where $\otimes$ is over R , and (2) $\sum c_i d_i = 1$. Such an element $\sum c_i \otimes d_i$ is called a separable idempotent for S . When R is contained in the center of S , S is called a separable R -algebra. The separable R -algebra S is called an Azumaya R -algebra if R is the center of S .

3. SEPARABLE QUATERNION ALGEBRAS.

Throughout, we assume that B is a commutative ring with 1 , and G ($= \{\sigma\}$) an automorphism group of order 2 of B , and that $B[j]$ is the generalized quaternion algebra over A , where A is the subring of elements fixed under σ . Our main goal in the section is to study a separable extension $B[j]$ over B without the assumption that 2 is a unit in A . We begin with a description of the set of separable idempotents for $B[j]$ (if there are any) over B . Clearly, $\{1 \otimes 1, 1 \otimes j, j \otimes 1, j \otimes j\}$ is a basis for $B[j] \otimes_B B[j]$.

LEMMA 3.1. The element $x = a_{11}(1 \otimes 1) + a_{12}(1 \otimes j) + a_{21}(j \otimes 1) + a_{22}(j \otimes j)$ is a separable idempotent for $B[j]$ over B if and only if (1) $a_{22} = -\sigma(a_{11})$ such that $\text{Tr}(a_{11}) = 1$, and (2) $a_{21} = \sigma(a_{12})$ such that $a_{12}((b - \sigma(b))) = 0$ for all b in B and $\text{Tr}(a_{12}) = 0$.

PROOF. Let x be a separable idempotent for $B[j]$ over B . Then $xu = ux$ for each u in $B[j]$. Hence $xj = jx$; that is,

$$\sigma(a_{11})(j \otimes 1) + \sigma(a_{12})(j \otimes j) - \sigma(a_{21})(1 \otimes 1) - \sigma(a_{22})(1 \otimes j) =$$

$a_{11}(1 \otimes j) - a_{12}(1 \otimes 1) + a_{21}(j \otimes j) - a_{22}(j \otimes 1)$. Equating corresponding coefficients, we have $\sigma(a_{11}) = -a_{22}$, $a_{12} = \sigma(a_{21})$; that is, $a_{22} = -\sigma(a_{11})$ and $a_{21} = \sigma(a_{12})$ for $\sigma^2 = 1$. Also, $bx = xb$ for all b in B , so $b_{12}(b - \sigma(b)) = 0$. Thus $x = a_{11}(1 \otimes 1) + a_{12}(1 \otimes j) + \sigma(a_{12})(j \otimes 1) - \sigma(a_{11})(j \otimes j)$ with $a_{12}(b - \sigma(b)) = 0$. Moreover, by the second condition of a separable idempotent, $a_{11} + (a_{12} + \sigma(a_{12}))j + \sigma(a_{11}) = 1$, so $\text{Tr}(a_{11}) = 1$ and $\text{Tr}(a_{12}) = 0$. Conversely, it is straightforward to verify that any x satisfying all equations as given is a separable idempotent.

THEOREM 3.2. $B[j]$ is a separable extension over B if and only if there is an element c in B such that $\text{Tr}(c) = 1$.

PROOF. The necessity is a consequence of Lemma 3.1. For the sufficiency, if $\text{Tr}(c) = 1$, we take $a_{11} = c$, $a_{12} = a_{21} = 0$. Then $a_{11}(1 \otimes 1) - \sigma(a_{11})(j \otimes j)$ is a separable idempotent for $B[j]$ by Lemma 3.1. Thus $B[j]$ is a separable extension over B .

Using Theorem 3.2, we can obtain a characterization of a separable extension $B[j]$ over B when $\text{Char}(A) = 2$.

THEOREM 3.3. Assume $\text{Char}(A) = 2$. Then, $B[j]$ is a separable extension over B if and only if B is a Galois extension over A .

PROOF. Let B be a Galois extension over A . Corollary 1.3 on P. 85 in [3] implies that $\text{Tr}(c) = 1$ for some c in B . Thus $B[j]$ is a separable extension over B by Theorem 3.2. Conversely, by Theorem 3.2 again, there exists an c in B such that $\text{Tr}(c) = 1$, so $(c + \sigma(c)) = 1$. By hypothesis, $\text{Char}(A) = 2$, $\sigma(c) = \sigma(-c) = -\sigma(c)$, so $c - \sigma(c) = 1$. Hence the ideal generated by $\{(b - \sigma(b)) / b \text{ in } B\} = B$. This implies that B is Galois over A by the statement 5 in Proposition 1.2 on P. 81 in [3].

Let us recall that the theorem of Parimala and Sridharan (Proposition 1.1 in [5]): Assume 2 is a unit in A . Then, B is Galois over A

if and only if $B \otimes_A B[j] \cong M_2(B)$, a matrix algebra over B of order 2.

We are going to improve it without the assumption that 2 is a unit in A .

THEOREM 3.4. If B is Galois over A , then $B \otimes_A B[j] \cong M_2(B)$.

PROOF. If B is Galois over A , there exists an c in B such that $\text{Tr}(c) = 1$ ([3], Corollary 1.3, P. 85). Hence $B[j]$ is a separable extension over A by Theorem 3.2. But B is also a separable extension over A by Proposition 1.2 in [3], so the transitive property of separable extensions ([4], Proposition 2.5) implies that $B[j]$ is a separable A -algebra. Moreover, we claim that (1) $B[j]$ is an Azumaya algebra over A , and (2) B is a maximal commutative subalgebra of $B[j]$. The proof of these facts was given in [7]. For completeness, we give an outline here. For part (1), it suffices to show that A is the center of $B[j]$. Clearly, A is contained in the center. Now, let $b + b'j$ be in the center. Then $j(b + b'j) = (b + b'j)j$ and $c(b + b'j) = (b + b'j)c$ for each c in B . Equating coefficients of the basis $\{1, j\}$ in the above equations, we have that b is in A and $b' = 0$ by Statement 5 in Proposition 1.2 on P. 81 in [3]. For part (2), to show that B is a maximal commutative subalgebra of $B[j]$ is to show that the commutant of B in $B[j]$ is B . The computation is similar to part (1).

Moreover, noting that B is separable over A , we then conclude that $B \otimes_A (B[j])^0 \cong \text{Hom}_B(B[j], B[j])$ by Theorem 5.5 on P. 65 in [3], and this implies that $B \otimes_A B[j] \cong M_2(B)$, where $(B[j])^0$ is the opposite ring.

In [7], the sufficiency of the Parimala and Sridharan theorem was shown by a different method from [5]. Now we slightly improve the statement without the assumption that 2 is a unit in A .

THEOREM 3.5. Let $B[j]$ be a separable extension over B . If $B \otimes_A B[j] \cong M_2(B)$, then B is Galois over A .

PROOF. Since $B[j]$ is a separable extension over B , there exists an element c in B such that $\text{Tr}(c) = 1$ by Theorem 3.2. Hence the sequence $B \rightarrow A \rightarrow 0$ is exact under the trace map. But A is projective over A , so the sequence splits, and then A is an A -direct summand of B . By hypothesis, $B \otimes_A B[j] \cong M_2(B)$ which is an Azumaya B -algebra, so $B[j]$ is an Azumaya A -algebra ([3], Corollary 1.10, P. 45). Therefore B is Galois over A by using the same argument as given in [7].

In Theorem 3.5, the hypothesis that $B \otimes_A B[j] \cong M_2(B)$ can be replaced by that $B \otimes_A B[j]$ is an Azumaya B -algebra with the same proof.

4. SPECIAL SEPARABLE QUATERNION ALGEBRAS.

Theorem 3.5 tells us that $B[j]$ is an Azumaya A -algebra such that $B \otimes_A B[j] \cong M_2(B)$ when B is Galois over A . In this section, we are going to discuss generalized quaternion algebras $B[j]$ in which 2 is a unit in A when B is projective and separable over A . With a similar argument as given in Lemma 3.1, we have

LEMMA 4.1. The element $a_{11}(1 \otimes 1) + a_{12}(1 \otimes j) + a_{21}(j \otimes 1) + a_{22}(j \otimes j)$ in $A[j] \otimes_A A[j]$ is a separable idempotent for $A[j]$ if and only if (1) $a_{22} = -a_{11}$ such that $2a_{11} = 1$, and (2) $a_{21} = a_{12}$ such that $2a_{12} = 0$.

THEOREM 4.2. The A -algebra $A[j]$ is separable if and only if 2 is a unit in A .

PROOF. The necessity is clear by Lemma 4.1; the sufficiency is immediate because $(1/2)(1 \otimes 1 - j \otimes j)$ is a separable idempotent.

Now we give a characterization of $B[j]$ in which 2 is a unit when B is projective and separable over A .

THEOREM 4.3. Let B be separable and projective over A . Then, $B[j]$ is a separable extension over B and projective over $A[j]$ as a bi-module if and only if 2 is a unit in A .

PROOF. Let 2 be a unit in A and let c be $(1/2)$. Then $\text{Tr}(c) = 1/2 + 1/2 = 1$, and hence $B[j]$ is separable over B by Theorem 3.2. By hypothesis, B is projective over A , so $B[j]$ is left projective over A (for $B[j]$ is left projective over B). Hence $B[j]$ is left projective over $A[j]$ ([3], Proposition 2.3, P. 48). We next claim that $B[j]$ is also right projective over $A[j]$. In fact, $\alpha: B \otimes_A A[j] \rightarrow B[j]$ defined by $\alpha(b \otimes 1 + b' \otimes j) = b \otimes b'j$ for all b and b' in B is an isomorphism as right $A[j]$ -modules. But B is projective over A , so $B \otimes_A A[j]$ is right projective over $A[j]$. This proves that $B[j]$ is right projective over $A[j]$. Thus $B[j] \otimes_A (B[j])^0$ is projective as $A[j]$ - $A[j]$ -module. Since $B[j]$ is a direct summand of $B[j] \otimes_A (B[j])^0$ as a $B[j] \otimes_A (B[j])^0$ -module (for $B[j]$ is separable over A), $B[j]$ is projective as a $A[j]$ - $A[j]$ -module.

Conversely, to show that 2 is a unit in A , it suffices to show that $A[j]$ is a separable A -algebra by Theorem 4.2. Since $B[j]$ is a separable extension over B , $\text{Tr}(c) = 1$ for some c in B by Theorem 3.2. Hence $\text{Tr}: B \rightarrow A \rightarrow 0$ is exact. We claim that Tr induces an exact sequence: $B[j] \rightarrow A[j] \rightarrow 0$ as $A[j]$ - $A[j]$ -modules. We define $\beta: B[j] \rightarrow A[j] \rightarrow 0$ by $\beta(b + b'j) = \text{Tr}(b) + \text{Tr}(b')j$. Clearly, β is an additive group homomorphism. Moreover, for a, a' in A , $(b + b'j)(a + a'j) = (ba - b'a') + (ba' + b'a)j$, so $\beta((b + b'j)(a + a'j)) = \text{Tr}(ba - b'a') + \text{Tr}(ba' + b'a)j = (a\text{Tr}(b) - a'\text{Tr}(b')) + (a'\text{Tr}(b) + a\text{Tr}(b'))j$. Also, $\beta(b + b'j)(a + a'j) = (\text{Tr}(b) + \text{Tr}(b')j)(a + a'j) = \beta((b + b'j)(a + a'j))$. Thus β is a right $A[j]$ -homomorphism. Similarly, by noting that $\text{Tr} = 1 + \epsilon$ and that $(\text{Tr})\epsilon = \text{Tr} = \epsilon(\text{Tr})$, it is straightforward to verify that β is a left $A[j]$ -homomorphism. But then $A[j]$ is $A[j]$ - $A[j]$ projective such that β is onto (for $\text{Tr}(c) = 1$ in $A[j]$). This implies that the exact

sequence $\beta: B[j] \rightarrow A[j] \rightarrow 0$ splits as $A[j]$ - $A[j]$ -modules. Thus $A[j]$ is an $A[j]$ -direct summand of $B[j]$. Now by hypothesis, $B[j]$ is $A[j]$ -projective, so $B[j] \otimes_A (B[j])^0$ is $A[j] \otimes_A A[j]$ -projective, where $(B[j])^0$ is the opposite algebra of $B[j]$. By hypothesis again, $B[j]$ is separable over A , so $B[j]$ is projective over $A[j] \otimes_A A[j]$. Therefore, the $A[j]$ -direct summand $A[j]$ of $B[j]$ is also projective over $A[j] \otimes_A A[j]$. This proves that $A[j]$ is separable over A , and so 2 is a unit in A by Theorem 4.2.

5. EXAMPLES.

This section includes several examples to illustrate our results.

(1) Let Z be the ring of integers, and $Z \times Z (= B)$ the ring of direct product of Z under the componentwise operations. Define $\sigma: Z \times Z \rightarrow Z \times Z$ by $\sigma(a, a') = (a', a)$ for a, a' in Z . Then σ is an automorphism group of order 2 and $\{(a, a) / a \text{ in } Z\} (= A)$ is the subring of $Z \times Z$ of the fixed elements under σ . Imbed Z in $Z \times Z$ by $a \rightarrow (a, a)$. Then we have

- (a) $Z \times Z$ is a free A -module with a basis $\{(1, 0), (0, 1)\}$.
- (b) $Z \times Z$ is separable over Z .
- (c) $(Z \times Z)[j]$ is a separable extension over $Z \times Z$ because $\text{Tr}((1, 0)) = (1, 0) + (0, 1) = (1, 1)$ by Theorem 3.2.

(d) $Z[j]$ is not separable over Z because 2 is not a unit in Z by Theorem 4.2.

(e) $(Z \times Z)[j]$ is not projective over $Z[j]$ because 2 is not a unit in Z by Theorem 4.3.

(2) Let $Z_{(3)}$ be the local ring of Z at the prime ideal (3) . Replace Z by $Z_{(3)}$ in Example (1). Then we have

- (a) 2 is a unit in $Z_{(3)}$.
- (b) All properties (a), (b) and (c) in Example (1) hold.

(c) $(Z_{(3)} \times Z_{(3)})[j]$ is projective over $Z_{(3)}[j]$ by Theorem 4.3.

(3) $Z \times Z$ and $Z_{(3)} \times Z_{(3)}$ in Example (1) and Example (2) are Galois over Z and $Z_{(3)}$ respectively by using Proposition 1.2 on P. 64 in [3]. Since $\text{Tr}((3, -2)) = (3, -2) + (-2, 3) = (1, 1)$ which is not in any maximal ideal of $Z \times Z$ or $Z_{(3)} \times Z_{(3)}$. Thus $(Z \times Z) \otimes_Z (Z \times Z)[j] \cong M_2(Z \times Z)$ and $(Z_{(3)} \times Z_{(3)}) \otimes_{Z_{(3)}} (Z_{(3)} \times Z_{(3)})[j] \cong M_2(Z_{(3)} \times Z_{(3)})$ by Theorem 3.4.

(4) Let i be the usual imaginary unit. Then $Z[i]$ is not separable over Z . $Z[i]$ has an automorphism group $\{\sigma: \sigma(a+bi) = a-bi \text{ for } a, b \text{ in } Z\}$ such that $\sigma^2 = 1$ and Z is the fixed ring of σ . Also, (a) $(Z[i])[j]$ is not separable over $Z[i]$, and (b) $Z[i]$ is not Galois over Z .

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