RINGS WITH INVOLUTION WHOSE SYMMETRIC ELEMENTS ARE CENTRAL

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(Received May 30, 1978 and in revised form February 6, 1979)

ABSTRACT. In a ring R with involution whose symmetric elements S are central, the skew-symmetric elements K form a Lie algebra over the commutative ring S . The classification of such rings which are 2-torsion free is equivalent to the classification of Lie algebras K over S equipped with a bilinear form f that is symmetric, invariant and satisfies $[[x,y],z] = f(y,z)x - f(z,x)y$. If S is a field of char $\neq 2$, $f \neq 0$ and $\dim K > 1$ then K is a semisimple Lie algebra if and only if f is nondegenerate. Moreover, the derived algebra K' is either the pure quaternions over S or a direct sum of mutually orthogonal abelian Lie ideals of $\dim \leq 2$.

KEY WORDS AND PHRASES. Ring with involution, symmetric and skew-symmetric elements, Lie algebra, symmetric and invariant bilinear form, Cartan's Criterion of semisimplicity of Lie algebras, pure quaternions, mutually orthogonal abelian Lie ideals.

1980 Mathematics Subject Classification Codes: 16A68, 16A28, 17B05.

1. INTRODUCTION and MAIN RESULTS.

Let R be a ring with an involution $*$, i.e., a map $R \rightarrow R$ such that for all $a, b \in R$

$$(a + b)^* = a^* + b^*, \quad (ab)^* = b^*a^* \quad \text{and} \quad a^{**} = a.$$

The sets of symmetric and skew-symmetric elements of R are respectively

$$S = \{a \in R \mid a^* = a\}, \quad K = \{a \in R \mid a^* = -a\}.$$

As usual, $[x, y] = xy - yx$ denotes the commutator of $x, y \in R$ and the symbol Z denotes the center of R .

If the symmetric elements of R are central, i.e., $S \subset Z$, then for abbreviation, R is called a CS-ring.

For all $x \in R$, $2x = x + x^* + x - x^*$ with $x + x^* \in S$, $x - x^* \in K$ and thus $2R \subset S + K$. If R is 2-torsion free then $S \cap K = 0$ and hence $\frac{1}{2} \in R$ implies that R is a group direct sum $S \oplus K$. If, additionally, R is a CS-ring then for $a \in S$, $x \in K$, $ax = xa = -(ax)^* \in K$ and therefore K is a Lie algebra over the commutative ring S with respect to commutation.

We have the following converse:

THEOREM 1. If S is a commutative ring, K is a 2-torsion free Lie algebra over S and $f: K \times K \rightarrow S$ is an S -bilinear map such that

- (1) $f(x, y) = f(y, x)$ (f is symmetric)
- (2) $f(x, [y, z]) = f([x, y], z)$ (f is invariant)
- (3) $[[x, y], z] = f(y, z)x - f(z, x)y$

then the group direct sum $R = S \oplus K$ can be made into a CS-ring by defining the multiplication and the involution, for all $a, b \in K$, as follows:

- (4) $(a + x)(b + y) = ab + f(x, y) + ay + bx + [x, y]$
- (5) $(a + x)^* = a - x.$

PROOF. Let $a, b, c \in S$ and $x, y, z \in K$. Multiplication in R is associative because

$$\begin{aligned}
& ((a+x)(b+y))(c+z) - (a+x)((b+y)(c+z)) \\
&= f(x,y)z + f([x,y],z) + [[x,y],z] - f(y,z)x - f(x,[y,z]) - [x,[y,z]] \\
&= f(x,y)z - f(y,z)x - [[z,x],y] \quad \text{by (2) and Jacobi identity} \\
&= 0 \quad \text{by (3)}.
\end{aligned}$$

From (1), (4) and (5),

$$((a+x)(b+y))^* = (b+y)^*(a+x)^*.$$

Hence, $*$ is an involution in R .

Since K is 2-torsion free, S and K are precisely the symmetric and skew-symmetric elements of R and therefore R is a CS-ring. ■

A CS-ring will, of course, satisfy identities (1) - (3) if we put $f(x,y) = 2(xy + yx)$ for all $x, y \in K$.

Note that if K has an S -basis and f is the dot product then (2) is the triple dot product and (3) is the "triple cross product" with opposite sign, that is, $[[x,y],z] = z \times (x \times y)$. We must, however, recall that the cross product of vectors is valid only for dimension ≤ 3 and it can also be ([3,p.61] or [5]) that CS-rings satisfy the standard polynomial of degree 4.

An example of a CS-ring is a ring of quaternions Q over a 2-torsion free commutative ring S , where Q admits an S -basis $1, i, j, ij$ such that given $a, b \in S$

$$i^2 = a, j^2 = b, ij = -ji$$

$$\text{and } i^* = 1, i^* = -i, j^* = -j.$$

The skew-symmetric part K of Q is a Lie algebra (with respect to commutation) of pure quaternions.

Henceforth, we shall tacitly assume that K is a Lie algebra over a field F of char $\neq 2$ and that K is equipped with an F -bilinear form f satisfying identities (1) - (3) such that $R = F \oplus K$ is a CS-ring with the multiplication and the involution defined according to (4) and (5).

As usual, the derived algebra and the radical of K are respectively $K' = [K, K]$, $K^\perp = \{x \in K \mid f(x, K) = 0\}$. A Lie ideal I of K is a subspace of K with $[I, K] \subset I$. It can be verified that a Lie ideal of K contained in $K^\perp$ is a proper ideal of R .

If $x, y \in K$ then $\langle x, y \rangle$ shall denote the F -subspace of K generated by x and y .

PROPOSITION 1. If $\dim K \neq 3$ then K' is abelian.

PROOF. We may assume $\dim K > 3$ since the prop. is trivially true for dimension < 3 . From (3), we have $[[x, y], z] \in \langle x, y \rangle$ for all x, y, z in K . Thus, if x, y, z, w are linearly independent vectors of K then

$$[[x, y], [z, w]] \in \langle x, y \rangle \cap \langle z, w \rangle = 0.$$

If $w \in \langle x, y, z \rangle$ where x, y, z are linearly independent then we choose a vector v in K such that $v \notin \langle x, y, z \rangle$ and thus $w + v \notin \langle x, y, z \rangle$. Consequently, $0 = [[x, y], [z, w + v]] = [[x, y], [z, w]]$.

Continuing this argument, we obtain $[[x, y], [z, w]] = 0$ for arbitrary x, y, z, w in K and thus $[K', K'] = 0$. ■

PROPOSITION 2. If $f \neq 0$ then $K^\perp$ is an ideal of R contained in K' and $\dim K/K' = 0$ or 1 .

PROOF. If $z \in K^\perp$ then by (2), $f(x, [y, z]) = f([x, y], z) = 0$ for all $x, y \in K$ and thus $[K, z] \subset K^\perp$. Hence, $K^\perp$ is a Lie ideal of K and an ideal of R . Since $f \neq 0$ there is a nonzero vector y in K with $f(y, y) \neq 0$. If $z \in K^\perp$ then by (3), $[[z, y], y] = f(y, y)z$ and thus $z = f(y, y)^{-1}[[z, y], y] \in K'$. Hence, $K^\perp \subset K'$.

If $\dim K/K' > 1$ then let x, y be vectors in K which are linearly independent modulo K' . By (3), $[[x, y], K] \in \langle x, y \rangle \cap K' = 0$ which forces $x \in K^\perp$. Since $K^\perp \subset K'$ we have $x \in K'$, a contradiction. Hence, $\dim K/K' = 0$ or 1 .

Putting $K' = 0$, we have

COROLLARY 1. If K is abelian and $\dim K > 1$ then $f = 0$ and $xy = 0$ for

all $x, y \in K$.

COROLLARY 2. If $K' \neq 0$ then $\dim K/K' > 1 \iff f = 0 \iff [K', K] = 0$.

PROOF. The second equivalence follows from (3). If $f \neq 0$ then by prop. 2, $\dim K/K' \leq 1$.

Conversely, if $f = 0$ then let x, y be vectors in K with $[x, y] \neq 0$ and thus $x, y \notin K'$. It suffices to show that the images $\underline{x}, \underline{y}$ in K/K' are linearly independent over F . Indeed, if $\underline{y} = a\underline{x}$ for some $a \in F$ then $y - ax \in K'$ and thus $0 = [x, y - ax] = [x, y]$, a contradiction. Hence, $\dim K/K' > 1$. ■

For the Lie algebras that we are considering there is a simple proof of Cartan's criterion for semisimplicity.

THEOREM 2. If $f \neq 0$ and $\dim K > 1$ then K has no nonzero abelian Lie ideals if and only if f is nondegenerate.

PROOF. If $K^\perp \neq 0$ then by prop. 2, $K^\perp$ is a nonzero Lie ideal contained in K' . By (3), $[K', K^\perp] = 0$ and hence $K^\perp$ is abelian.

Conversely, if K has a nonzero abelian Lie ideal I then let y, z be nonzero vectors in I and x be any vector of K such that x and y are linearly independent. By (3), $f(y, z)x - f(z, x)y = [[x, y], z] \in [I, I] = 0$ and thus $f(y, z) = f(z, x) = 0$. Hence, $f(z, K) = 0$ and $K^\perp \neq 0$. ■

THEOREM 3. If $f \neq 0$ then K' is either a Lie algebra of pure quaternions over F or a direct sum of mutually orthogonal abelian Lie ideals of K with $\dim \leq 2$.

PROOF. We may assume $K' \neq 0$ for otherwise, prop. 2 would imply that K is of dim 1. We have only to consider the two cases, $\dim K/K' = 0, 1$.

Suppose $K' = K$. Since $[K', K'] = [K, K] \neq 0$, $\dim K = 3$ by prop. 1. If $K \neq 0$ then let $K = K^\perp \oplus V$ where $\dim V \leq 2$ and by (3), $0 = [K', K^\perp] = [K, K^\perp]$ which implies that $K' = [V, V]$ is of $\dim \leq 1$, contradictory to $K' = K$.

Hence, $K^\perp = 0$. Since the bilinear form f is symmetric and nondegenerate, K has an orthogonal basis x, y, z . As $K' = K$, the commutators $[x, y]$, $[y, z]$, $[z, x]$ also form a basis of K . By (2), $f(x, [x, y]) = f(y, [x, y]) = 0$ and hence $[x, y]$ is orthogonal to x and y . Consequently, $[x, y] = z$, $[y, z] = ax$, $[z, x] = by$ where $a, b \in F$. We can now easily derive from (3) that $f(x, x) = -b$, $f(y, y) = -a$ and $f(z, z) = -ab$. Hence, $K' = K$ is a Lie algebra of pure quaternions over F .

Suppose $\dim K/K' = 1$. We have $K^\perp \subset K'$ by prop. 2. To show $K' \subset K^\perp$, let $x \in K'$ and choose $0 \neq y \notin K'$. By (3), $f(y, z)x - f(z, x)y = [[x, y], z] \in K'$ for all $z \in K$ and thus $f(K, x) = 0$. Hence, $x \in K^\perp$ and $K^\perp = K'$. Moreover, $0 = [K', K^\perp] = [K', K']$. Since $f \neq 0$, there exists a nonzero vector $e \in K/K'$ with $f(e, e) \neq 0$. Let $d(x) = [x, e]$ for all $x \in K'$. By (3), $d^2(x) = [[x, e], e] = f(e, e)x$ and hence $d^2 = f(e, e)I$ where I is the identity map of K' . Since every nonzero vector x and K' is in the d -invariant subspace $L_x = \langle x, d(x) \rangle$, it follows from [2, p.87] that K' is completely reducible as a module for d . Clearly, each L_x is an abelian Lie ideal of K with $\dim \leq 2$ and $f(L_x, L_y) = 0$ for $x \neq y$. ■

NOTE: This paper constitutes a part of the author's Ph.D. thesis written under the supervision of Professor D. Z. Djokovic at the University of Waterloo, Ontario, Canada.

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