GENERALIZED KÖTHE-TOEPLITZ DUALS

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ABSTRACT. The α and β-duals spaces of generalized ℓ_p spaces are characterized, where 0 < p ≤ ∞. The question of when the α and β dual spaces coincide is also considered.

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1. INTRODUCTION.

X and Y denote complex Banach spaces with zero elements 0, and ||.|| denotes the norm in either X or Y. The continuous dual of X is written X*. By s(X) we mean the space of all X-valued sequences x = (x_k), where x_k ∈ X for k ∈ N = {1,2,3,...}.
If $0 < p < \infty$, we mean by $\ell_p(X)$ the space of all $X$-valued sequences $x = (x_k)$ such that $\sum |x_k|^p < \infty$. Sums are over $k \in \mathbb{N}$, unless otherwise indicated.

By $\ell_\infty(X)$ we denote the space of all $x = (x_k)$ such that $\sup |x_k| < \infty$.

In case $X = \mathbb{C}$, the space of complex numbers, we write $\ell_p$ instead of $\ell_p(\mathbb{C})$.

Let $A = (A_k)$ denote a sequence of linear, but not necessarily bounded, operators on $X$ into $Y$. If $E$ is any nonempty subset of $s(X)$ then the $\alpha$-dual of $E$ is defined to be

$$E^\alpha = \{ A : \sum |A_k x_k| < \infty, \text{ for all } x \in E \}. $$

The $\beta$-dual of $E$ is defined to be

$$E^\beta = \{ A : \sum A_k x_k \text{ converges, for all } x \in E \}. $$

Since $Y$ is complete, we have $E^\alpha \subseteq E^\beta$. The $\alpha$ and $\beta$ duals of $E$ may be regarded as generalized Köthe-Toeplitz duals, since in case $X = Y = \mathbb{C}$, when the $A_k$ may be identified with complex numbers $a_k$, the duals reduce to the classical spaces first considered by Köthe and Toeplitz [1].

Using the notation $(1/p) + (1/q) = 1$, where $1 \leq p \leq \infty$, with the convention that $q = \infty$ when $p = 1$, and $q = 1$ when $p = \infty$, it is well-known that

$$\ell_p^\alpha = \ell_p^\beta = \ell_q. $$

(1.1)

We shall see that, in general, $\ell_p^\alpha(X) \subset \ell_p^\beta(X)$, where the inclusion may be strict. However, when $0 < p \leq 1$ the $\alpha$ and $\beta$ duals coincide. Also, when $1 < p \leq \infty$, the $\alpha$ and $\beta$ duals coincide provided that $Y$ is finite dimensional.
2. CHARACTERIZATION OF THE DUALS.

THEOREM 1. Let $0 < p \leq 1$. Then $A \in \ell^p_p(X)$ if and only if there exists $m \in \mathbb{N}$ such that $A_k$ is bounded, for all $k \geq m$, and

$$H = \sup_{k \geq m} ||A_k|| < \infty.$$  \hspace{1cm} (2.1)

PROOF. Sufficiency. Let (2.1) hold and $\sum ||x_k||^p < \infty$. By a familiar inequality, see for example Maddox [2], page 22,

$$\left( \sum_{k=m}^{\infty} ||A_k x_k|| \right)^p \leq \sum_{k=m}^{\infty} ||A_k||^p ||x_k||^p \leq \sum_{k=m}^{\infty} ||A_k||^p ||x_k||^p \leq H^p \sum ||x_k||^p.$$  

Hence $\sum A_k x_k$ is absolutely convergent, and so convergent.

Necessity. Let $A \in \ell^p_p(X)$ and suppose, if possible, that no such $m$ exists. Then there are natural numbers $k(1) < k(2) < \ldots$ and $z_i \in X$, $||z_i|| \leq 1$, such that for $i \in \mathbb{N},$

$$||A_{k(i)} z_i|| > i^{2/p}. \hspace{1cm} (2.2)$$

Define $x_k = z_i i^{2/p}$ for $k = k(i)$ and $x_k = 0$ otherwise. Then $x \in \ell_p(X)$ since $\sum ||x_k||^p \leq \pi^2/6$, but $||A_k x_k|| > 1$ for infinitely many $k$, contrary to the fact that $\sum A_k x_k$ converges.

Now suppose, if possible, that $\sup_{k \geq m} ||A_k|| = \infty$. Then there are natural numbers $k(1) < k(2) < \ldots$ with $k(1) \geq m$ such that for $i \in \mathbb{N},$

$$||A_{k(i)} z_i|| > 2i^{2/p}. \hspace{1cm} (2.3)$$

Choose $z_i \in X$ with $||z_i|| \leq 1$ such that $2||A_{k(i)} z_i|| \geq ||A_{k(i)}||$, so by (2.3)
we see that (2.2) holds with the new $k(i)$ and $z_i$. We may define $x \in \ell_p(X)$ as above and obtain a contradiction. Hence (2.1) must hold, and the proof is complete.

If we examine the proof of Theorem 1 we see that in the sufficiency we had $\sum |A_k x_k| < \infty$, so that $A \in \ell_p^\alpha(X)$. Also, in the necessity, the constructions involved $x \in \ell_p(X)$ such that $\sum |A_k x_k|$ was divergent. Hence we have:

**THEOREM 2.** If $0 < p \leq 1$ then

$$\ell_p^\alpha(X) = \ell_p^\varphi(X).$$

Next we consider the case $1 < p < \infty$.

**THEOREM 3.** Let $1 < p < \infty$. Then $A \in \ell_p^\alpha(X)$ if and only if there exists $m \in \mathbb{N}$ such that $A_k$ is bounded for all $k \geq m$, and

$$M = \sum_{k=m}^{\infty} ||A_k||^q < \infty. \quad (2.4)$$

**PROOF.** Sufficiency. Let (2.4) hold and $x \in \ell_p(X)$. By Hölder's inequality,

$$\sum_{k=m}^{\infty} ||A_k x_k|| \leq M^{1/q} (\sum ||x_k||^p)^{1/p} < \infty.$$

Necessity. Since $\ell_p^\alpha(X) \subset \ell_1^\alpha(X)$ when $p > 1$, the existence of the $m$ in the theorem follows from Theorems 1 and 2.

Now for $k \geq m$ we may choose $z_k \in X$ with $||z_k|| \leq 1$ such that

$$2||A_k z_k|| \geq ||A_k||.$$

For all $\lambda \in \ell_p$ we have $(\lambda_k z_k) \in \ell_p(X)$, so
for all $\lambda \in \ell_p$. By (1.1) it follows that
\[ H = \sum_{k=m}^{\infty} \| A_k z_k \|^q < \infty, \]
whence $M \leq 2^q H$, so (2.4) holds, and the proof is complete.

**THEOREM 4.** Let $1 < p < \infty$. Then $A \in \ell_\beta^p(X)$ if and only if there exists $m \in \mathbb{N}$ such that $A_k$ is bounded for all $k \geq m$, and
\[ \sup_{k=m}^{\infty} \| A_k f \|^q < \infty, \tag{2.5} \]
where the supremum is over all $f \in \mathcal{Y}^*$ with $\| f \| \leq 1$.

**PROOF.** With the restriction that all the $A_k$ are bounded, and with different notation, this result was proved by Thorp [3]. Only the existence of $m$ in the necessity needs attention, and this follows from Theorems 1 and 2, and the fact that $\ell_\beta^p(X) \subset \ell_1^\beta(X)$.

Finally, we examine the case $p = \infty$. The proofs are left to the reader.

We remark that with the restriction that all the $A_k$ are bounded, the result concerning $\ell_\infty^\beta(X)$ was given by Maddox [4].

**THEOREM 5.** $A \in \ell_\infty^\alpha(X)$ if and only if there exists $m \in \mathbb{N}$ such that $A_k$ is bounded for all $k \geq m$, and
\[ \sum_{k=m}^{\infty} \| A_k \| < \infty. \tag{2.6} \]

**THEOREM 6.** $A \in \ell_\infty^\beta(X)$ if and only if there exists $m \in \mathbb{N}$ such that $A_k$ is bounded for all $k \geq m$, and
\[ \sup_{k=m}^{m+n} \left| \sum_{k=m}^{m+n} A_k x_k \right| < \infty, \quad (2.7) \]

\[ \sup_{k=m}^{m+n} \left| \sum_{k=m}^{m+n} A_k x_k \right| \to 0 \quad (m \to \infty), \quad (2.8) \]

where the suprema are over all \( n \geq 0 \) and all \( x_k \in X \) with \( \|x_k\| \leq 1 \).

3. **COINCIDENCE OF DUALS.**

It was shown in Theorem 2 that, when \( 0 < p \leq 1 \), \( \ell_p^\alpha(X) = \ell_p^\beta(X) \) for any Banach spaces \( X \) and \( Y \).

We next shown that, when \( 1 < p < \infty \), the inclusion \( \ell_p^\alpha(X) \subset \ell_p^\beta(X) \) may be strict.

**THEOREM 7.** If \( 1 < p < \infty \) then there are Banach spaces \( X \) and \( Y \) such that \( \ell_p^\alpha(X) \subset \ell_p^\beta(X) \) with strict inclusion.

**PROOF.** Take \( X = Y = \ell_p \) and write

\[ e_k = (0,0,\ldots,1,0,0,\ldots) \]

where 1 is in the \( k \)-place and there are zeros elsewhere. Define bounded linear operators \( A_k \) on \( \ell_p \) into itself by

\[ A_k x = x_k e_k \]

for each \( x = (x_k) \in \ell_p \). Then \( \|A_k\| = 1 \) for all \( k \in \mathbb{N} \), so \( A \) is not in \( \ell_p^\alpha(X) \) by Theorem 3.

Let us now show that (2.5) holds. Take any \( f \in \ell_p^* \) with \( \|f\| \leq 1 \). Then for \( x \in \ell_p \) we have

\[ f(x) = \sum_i f_i x_i \]
for some \( (f_i) \) such that \( \sum |f_i|^q \leq 1 \). Hence, by definition of \( A_k^* \),

\[
(A_k^* f)(x) = f(A_k x) = f_k x_k
\]

and so \( |A_k^* f| = |f_k| \). Hence

\[
\sum |A_k^* f|^q = \sum |f_k|^q \leq 1,
\]

so by Theorem 4 we have \( A \in \ell_p^\beta(X) \).

Still with the case \( 1 < p < \infty \) we have:

**Theorem 8.** If \( 1 < p < \infty \) and \( Y \) is finite dimensional then for any \( X \) we have

\[
\ell_p^\alpha(X) = \ell_p^\beta(X).
\]

**Proof.** We have to show that \( A \in \ell_p^\beta(X) \) implies \( A \in \ell_p^\alpha(X) \). Now if \( A \in \ell_p^\beta(X) \) then by Theorem 4 there exists \( m \in \mathbb{N} \) such that \( A_k \) is bounded for all \( k \geq m \). Suppose \( Y \) has finite dimension \( n \) and that \( (b_1, b_2, \ldots, b_n) \) is a Hamel base for \( Y \). Then \( y \in Y \) implies

\[
y = \sum_{i=1}^{n} \lambda_i(y)b_i
\]

where each \( \lambda_i \in Y^* \). Take \( z \in X \) and \( k \geq M \). Then

\[
A_kz = \sum_{i=1}^{n} \lambda_i(A_k z)b_i
\]

and \( \lambda_i \circ A_k \in X^* \). Since \( \sum_{k=m}^{\infty} A_k x_k \) converges for all \( x \in \ell_p(X) \) we have

\[
\sum_{k=m}^{\infty} (\lambda_i \circ A_k) x_k
\]

convergent for all \( x \in \ell_p(X) \) and each \( i \).
Choose $z_k \in X$, $||z_k|| \leq 1$ such that $2|\lambda_i \circ A_k|^q \geq \lambda_i \circ A_k||. 
 If $t \in \ell_p$ then $(t_k z_k) \in \ell_p(X)$ so that 
\[ \sum_{k=m}^\infty t_k (\lambda_i \circ A_k) z_k \]
converges for all $t \in \ell_p$, whence for each $i$, 
\[ \sum_{k=m}^\infty \lambda_i \circ A_k ||^q < \infty. \quad (2.10) \]

By (2.9) and Hölder's inequality, 
\[ ||A_k||^q \leq \sum_{i=1}^n ||\lambda_i \circ A_k||^q \cdot (\sum_{i=1}^n ||b_i||^p)^{q/p}. \quad (2.11) \]

Denoting the final term in (2.11) by $H$, 
\[ \sum_{k=m}^\infty ||A_k||^q \leq H \sum_{i=1}^n \sum_{k=m}^\infty ||\lambda_i \circ A_k||^q. \quad (2.12) \]

It follows from (2.10) and (2.12) that (2.4) holds, so by Theorem 3 we have $A \in \ell_p^a(X)$. 

For certain values of $p$, and any $X$, the next result is the converse of Theorem 8.

**Theorem 9.** If $2 < p < \infty$ and $\ell_p^a(X) = \ell_p^b(X)$ then $Y$ must be finite dimensional.

**Proof.** Suppose, if possible, that $Y$ is infinite dimensional. Since $q < 2$, if $c_k = k^{-2/q}$ then $\sum c_k < \infty$. By the Dvoretzky-Rogers theorem [5], there exists an unconditionally convergent series $\sum y_k$ in $Y$ such that $||y_k||^q = c_k$ for $k \in N$. Hence 
\[ \sum ||y_k||^q \] diverges. \quad (2.13)

Take $f \in X^*$ with $||f|| = 1$ and define rank one operators $A_k = y_k \otimes f$. 
Then $||A_k|| = ||y_k||$, so by (2.13) and Theorem 3, $A$ is not in $\ell_p^a(X)$. 

Now if \( x \in \ell_p(X) \) then

\[
\sum A_k x_k = \sum f(x_k) y_k.
\]

But \((f(x_k)) \in \ell_\infty\) and \(y_k\) is unconditionally convergent, so that \(\sum f(x_k) y_k\) converges, whence \(A \in \ell_\beta^p(X)\), which gives a contradiction.

We remark that it would appear that the argument of Theorem 9 cannot be used in the case \(p = 2\), since in a general Hilbert space \(Y\) the unconditional convergence of \(\sum y_k\) implies that \(\sum ||y_k||^2\).

However, we can deal with the case \(p = 2\) of Theorem 9 when \(Y\) is a Hilbert space:

**THEOREM 10.** Let \(Y\) be a Hilbert space and suppose \(\ell_2^\alpha(X) = \ell_2^\beta(X)\).
Then \(Y\) must be finite dimensional.

**PROOF.** Suppose, if possible, that \(Y\) is infinite dimensional. Choose an orthonormal sequence \((e_k)\) in \(Y\) and denote the inner product in \(Y\) by \((y_1, y_2)\).

Take \(g \in X^*, ||g|| = 1\) and define rank one operators \(A_k = e_k \otimes g\), so that \(||A_k|| = 1\). Now let \(f \in Y^*\) with \(||f|| \leq 1\). Then there exists \(y \in Y\) such that

\[
f(z) = (z, y)
\]

for all \(z \in Y\), with \(||y|| = ||f|| \leq 1\). Then for \(x \in X\),

\[
(A^*f)(x) = (g(x)e_k, y) = g(x)(e_k, y).
\]

Hence \(||A_k^*f|| \leq ||(e_k, y)||\), so by Bessel's inequality,

\[
\sum ||A_k^*f||^2 \leq ||y||^2 \leq 1.
\]

Thus (2.5) holds with \(q = 2\), and so \(A \in \ell_2^\beta(X)\). But \(A \notin \ell_2^\alpha(X)\) since \(||A_k|| = 1\) for all \(k\). This contradiction implies our result.
The case $p = \infty$ is due essentially to Thorp [3], who shows that $\ell_p^\alpha(X) = \ell_p^\beta(X)$ if and only if $Y$ is finite dimensional.

REFERENCES


