ON THE CONSISTENCY OF LIMITATION METHODS FOR (N, p_n) SUMMABLE SEQUENCES

NAND KISHORE
U.K. MISRA

Department of Mathematics
Berhampur University
Berhampur-760007
Orissa, INDIA

(Received November 8, 1978 and in revised form March 12, 1979)

ABSTRACT. Two limitation methods, A and B, are said to be consistent for a class b of sequences, iff, every sequence belonging to b is limitable both by A and B and that the A-limit equals the B-limit. Any two regular limitation methods are consistent for the class-c of convergent sequences. However, this is not true in general and in fact, corresponding to every bounded non-convergent sequence it is possible to determine two T-matrices such that they limit the sequence to two different values. In this paper, we establish the necessary and sufficient conditions for the consistency of two limitation methods, for (N, p_n) summable sequences.

KEY WORDS AND PHRASES. Summable sequence, Limitation methods, Infinite matrices.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES: 40A05.

1. INTRODUCTION.

Let $\{s_n\}$ be a sequence of real or complex terms. Let $A \equiv (a_{mn})$, $\infty \times \infty$, be an infinite matrix over a real or complex field. Then the transform, given by

$$T_m = \sum_{n=1}^{\infty} a_{mn} s_n, \quad (1.1)$$

if it exists for every m , is called the A-transform of the sequence $\{s_n\}$. If $\lim_{m \rightarrow \infty} T_m = s$, $\{s_n\}$ is said to be A-limitable to s . Moreover if $\lim_{n \rightarrow \infty} s_n = s$ implies $\lim_{m \rightarrow \infty} T_m = s$, the matrix A is said to be regular. In 1911, Toeplitz obtain-

ed necessary and sufficient conditions for a regular matrix as follows:

A Matrix $A \equiv (a_{mn})$ is regular, iff,

$$(i) \quad \sup_m \sum_n |a_{mn}| \leq M, \text{ an absolute constant;} \quad (1.2)$$

$$(ii) \quad \lim_{m \rightarrow \infty} a_{mn} = 0, \text{ for every fixed } n; \quad (1.3)$$

$$(iii) \quad \sum_{n=1}^{\infty} a_{mn} = A_m \rightarrow 1 \text{ as } m \rightarrow \infty. \quad (1.4)$$

Let T be a class of matrices satisfying the conditions (1.2) to (1.4). Any matrix of the class T is called a Toeplitz matrix or simply a T -matrix. Thus, a matrix A is regular if it belongs to the class T .

Let $\{p_n\}$ be a sequence of constants, real or complex, such that

$P_m = (p_0 + p_1 + \dots + p_m) \neq 0$, for any $m = 0, 1, 2, \dots$. Then the limitation method for which

$$a_{mn} = \begin{cases} \frac{p_{m-n}}{p_m}, & \text{for } n \leq m \\ 0, & \text{for } n > m \end{cases} \quad (1.5)$$

is called the Nörlund method, or simply (N, p_n) method. A (N, p_n) method is regular, iff,

$$(i) \quad \sum_{n=0}^m |p_n| = O(|P_m|), \text{ for all } m, \quad (1.6)$$

$$(ii) \quad \lim_{m \rightarrow \infty} \frac{p_m}{P_m} = 0. \quad (1.7)$$

We use the following notations:

$$(i) \quad p(x) = \sum p_n x^n; \quad (1.8)$$

$$(ii) \quad \frac{1}{p(x)} = \sum c_n x^n; \quad (1.9)$$

$$(iii) \quad \{p_n\} \in M, \text{ iff, } p_0 = 1, p_n > 0 \text{ and } p_{n+1}^2 \leq p_n p_{n+2}; \quad (1.10)$$

$$(iv) \quad t_m = \frac{1}{p_m} \sum_{n=0}^m p_{m-n} s_n, p_m \neq 0; \quad (1.11)$$

(v) Throughout the paper, M is taken for an absolute constant not necessarily the same at each occurrence.

2. MAIN RESULTS.

Two limitation methods, A and B , are said to be consistent for a class b of

sequences, iff, every sequence belonging to b is limitable both by A and B , and the A -limit is equal to the B -limit. Thus, any two matrix methods, generated by the matrices of the class T , are consistent for the class- c of convergent sequences. However, this is not true in general, and in fact, corresponding to every bounded non-convergent sequence it is always possible to determine two T -matrices such that they limit the sequence to two different values (see Cooke [1], page 97).

A limitation method Q is said to include a limitation method P if every sequence limitable by P is limitable by Q and to the same limit. Sometimes we indicate this by set theoretic inclusion as, $P \subseteq Q$, meaning thereby that space of sequences limitable by Q includes that limitable by P .

Two limitation methods, determined by the matrices $A \equiv (a_{mn})$ and $B \equiv (b_{mn})$ are said to be equivalent for a class b of sequences, iff, for every $\{s_n\} \in b$

$$\lim_{m \rightarrow \infty} (T_m - T_m^1) = 0, \quad (2.1)$$

where

$$T_m = \sum_{n=1}^{\infty} a_{mn} s_n \text{ and } T_m^1 = \sum_{n=1}^{\infty} b_{mn} s_n.$$

Hence two limitation methods A and B are consistent for a class b of sequences iff,

$$(i) \quad A \text{ and } B \text{ are equivalent, for class } b, \quad (2.2)$$

$$(ii) \quad A \text{ and } B \text{ limit every sequence } \in b. \quad (2.3)$$

Let b be the class of all sequences that are (N, p_n) summable. To ensure condition (2.2), we prove the following theorems in section 3.

THEOREM 1. Let (N, p_n) be a regular Nörlund method and let $\{p_n\} \in M$. The necessary and sufficient conditions that any two limitation methods A and B , determined respectively by the matrices (a_{mn}) and (b_{mn}) , are equivalent for all such (N, p_n) summable sequences are

$$(i) \quad \lim_{m \rightarrow \infty} \gamma_{mn} = 0, \text{ for all fixed } n; \quad (2.4)$$

$$(ii) \quad \lim_{m \rightarrow \infty} \sum_n \gamma_{mn} = 0; \quad (2.5)$$

$$(iii) \quad \sum_{k=0}^{\infty} p_k \left| \sum_{n=k}^{\infty} \gamma_{mn} c_{n-k} \right| \leq M, \text{ for every } m, \quad (2.6)$$

where

$$(a_{m n}) - (b_{m n}) = (\gamma_{m n}).$$

Condition (2.3) implies that $(N, p_n) \subseteq A$ and $(N, p_n) \subseteq B$. In section 4, we prove the following theorem:

THEOREM 2. Let (N, p_n) be a regular Nörlund method and let $\{p_n\}_1 \in M$. Then the limitation method A , determined by the matrix $A \equiv (a_{m n})$, $\infty \times \infty$, belonging to the class T , includes (N, p_n) , iff,

$$\sup_m \sum_{k=0}^{\infty} p_k \left| \sum_{n=k}^{\infty} a_{m n} c_{n-k} \right| \leq M, \quad (2.7)$$

where c_k is as defined in (1.9).

We required the following lemma of Kaluza (see Hardy [2], page 68) in proving our theorem.

LEMMA. If $p(x) = \sum p_n x^n$ is convergent for $|x| < 1$, and $\{p_n\}_1 \in M$, and further

$$p(x)^{-1} = 1 + c_1 x + c_2 x^2 + \dots,$$

then

$$\sum |c_n| \leq 2. \text{ If } \sum p_n = \infty, \text{ then } \sum_{n=1}^{\infty} |c_n| = 2.$$

3. PROOF OF THEOREM 1.

At the outset we observe that if (N, p_n) is regular and $p_n \geq 0$, for each n , then in view of the regularity condition, $\lim_{n \rightarrow \infty} \frac{p_{n+1}}{p_n} = 1$ and the series $\sum p_n x^n$ is absolutely convergent for $|x| < 1$, as such the series

$$\sum_{n=0}^{\infty} p_n x^n (1-x), \quad (3.1)$$

is also so, for $|x| < 1$. But then the series (3.1) equals $\sum p_n x^n$, and accordingly the series $\sum p_n x^n$ is absolutely convergent for $|x| < 1$.

Now we lay down the proof.

(If part): We have

$$t_n = \frac{1}{p_n} \sum_{k=0}^n p_{n-k} s_k.$$

Then, for $|x| < 1$, we have

$$\sum_{n=0}^{\infty} t_n p_n x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n p_{n-k} s_k \right) x^n = p(x) s(x).$$

Hence

$$\begin{aligned} s(x) &= \frac{1}{p(x)} \sum_{n=0}^{\infty} t_n p_n x^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n t_k p_k c_{n-k} \right) x^n \end{aligned}$$

Equating the coefficients of x^n , from both sides, we have

$$s_n = \sum_{k=0}^n t_k p_k c_{n-k}. \quad (3.2)$$

Now

$$\begin{aligned} \sum_{n=0}^{\infty} \gamma_m n s_n &= \sum_{n=0}^{\infty} \gamma_m n \sum_{k=0}^n p_k t_k c_{n-k} \\ &= \sum_{k=0}^{\infty} t_k p_k \sum_{n=k}^{\infty} \gamma_m n c_{n-k} \end{aligned}$$

Let us put $t_k = t + \epsilon_k$, where $t = \lim_{k \rightarrow \infty} t_k$ and $\{\epsilon_k\}$ is a null sequence. Then

$$\begin{aligned} \sum_{n=0}^{\infty} \gamma_m n s_n &= t \sum_{k=0}^{\infty} p_k \sum_{n=k}^{\infty} \gamma_m n c_{n-k} + \sum_{k=0}^{\infty} (p_k \sum_{n=k}^{\infty} \gamma_m n c_{n-k}) \epsilon_k \\ &= t \sum_{k=0}^{\infty} \gamma_m n \sum_{k=0}^n p_k c_{n-k} + \sum_{k=0}^{\infty} (p_k \sum_{n=k}^{\infty} \gamma_m n c_{n-k}) \epsilon_k \\ &= t \sum_{k=0}^{\infty} \gamma_m n + \sum_{k=0}^{\infty} (p_k \sum_{n=k}^{\infty} \gamma_m n c_{n-k}) \epsilon_k. \end{aligned}$$

Taking the limit as $m \rightarrow \infty$ and making use of condition (2.5), we have

$$\begin{aligned} \lim_{m \rightarrow \infty} \sum_{n=0}^{\infty} \gamma_m n s_n &= \lim_{m \rightarrow \infty} \sum_{k=0}^{\infty} (p_k \sum_{n=k}^{\infty} \gamma_m n c_{n-k}) \epsilon_k \\ &= \lim_{m \rightarrow \infty} \sum_{k=0}^{\infty} d_{mk} \epsilon_k, \text{ say,} \end{aligned}$$

where

$$d_{mk} = p_k \sum_{n=k}^{\infty} \gamma_m n c_{n-k}$$

Hence the proof is completed if we show that

$$\lim_{m \rightarrow \infty} \sum d_{mk} \epsilon_k = 0.$$

But since $\{\epsilon_k\} \rightarrow 0$ as $k \rightarrow \infty$, it is sufficient to show

$$(i) \quad \sum_k |d_{mk}| < M, \text{ for every } m; \quad (3.3)$$

$$(ii) \quad \lim_{m \rightarrow \infty} d_{mk} = 0, \text{ for every fixed } k. \quad (3.4)$$

Here (3.3) follows from (2.6). For establishing (3.4), it is sufficient to show that

$$\lim_{m \rightarrow \infty} \left(\sum_{n=k}^{\infty} \gamma_m n c_{n-k} \right) = 0, \quad (3.5)$$

for every fixed k .

Now since $\sum |c_n|$ is convergent, for any arbitrary small $\epsilon > 0$, we can have n_0 such that

$$\sum_{n > n_0} |c_n| < \frac{\epsilon}{2M}, \quad (3.6)$$

where M is as specified in (2.6).

Further let

$$\sum_{n \leq n_0} |c_n| \leq A, \quad (3.7)$$

where A is a finite constant. In view of (2.4) and the fact that n_0 is a finite positive integer, we can have m_0 dependent on ϵ such that

$$|\gamma_{m,n}| < \frac{\epsilon}{2A}, \quad (3.8)$$

for all $m > m_0(\epsilon)$ and all $n=k, k+1, \dots, n_0 + k$. Also

$$\begin{aligned} |\gamma_{m,h}| &= \left| \sum_{n=h}^{\infty} \gamma_{m,n} \sum_{k=0}^{n-h} p_k c_{n-h-k} \right| \\ &= \left| \sum_{k=0}^{\infty} p_k \sum_{n=h+k}^{\infty} \gamma_{m,n} c_{n-h-k} \right| \\ &\leq \sum_{k=0}^{\infty} p_{k+h} \left| \sum_{n=h+k}^{\infty} \gamma_{m,n} c_{n-h-k} \right| \\ &\leq \sum_{v=h}^{\infty} p_v \left| \sum_{n=v}^{\infty} \gamma_{m,n} c_{n-v} \right| \\ &\leq M, \text{ by (2.6),} \end{aligned} \quad (3.9)$$

for all m and h . Hence, finally

$$\begin{aligned} \left| \sum_{n=k}^{\infty} \gamma_{m,n} c_{n-k} \right| &= \left| \sum_{n=0}^{\infty} \gamma_{m,n+k} c_n \right| \\ &\leq \left| \sum_{n=0}^{n_0} \gamma_{m,n+k} c_n \right| + \left| \sum_{n > n_0} \gamma_{m,n+k} c_n \right| \\ &\leq \frac{\epsilon}{2A} \sum_{n=0}^{n_0} |c_n| + M \sum_{n > n_0} |c_n|, \text{ for } m > m_0, \text{ by (3.8) and (3.9).} \\ &\leq \frac{\epsilon}{2A} A + M \frac{\epsilon}{2M}, \text{ by (3.7) and (3.6).} \\ &= \epsilon. \end{aligned} \quad (3.10)$$

Hence $\lim_{m \rightarrow \infty} d_{m,k} = 0$, for every fixed k , and the "if part" of the theorem is proved.

(Only if part):

Let the limitation method of A and B be equivalent for all (N, p_n) summable

sequences. Then

$$\lim_{m \rightarrow \infty} \sum_n \gamma_{mn} s_n = 0,$$

where $(a_{mn}) - (b_{mn}) = (\gamma_{mn})$, and $\{s_n\}$ is a (N, p_n) summable sequence.

We have

$$\sum_{n=0}^{\infty} \gamma_{mn} s_n = \sum_{k=0}^{\infty} (P_k \sum_{n=k}^{\infty} \gamma_{mn} c_{n-k}) t'_k.$$

As $\{s_n\}$ is a (N, p_n) summable sequence,

(i) take $\{s_n\} = \{\delta_n^h\}$, so that $t_k = \frac{p_{k-h}}{p_k}$, for every $k > h$.

Then

$$\begin{aligned} \sum_n \gamma_{mn} s_n &= \sum_{k=0}^{\infty} p_{k-h} \sum_{n=k}^{\infty} \gamma_{mn} c_{n-k} \\ &= \sum_{n=0}^{\infty} \gamma_{mn} \sum_{k=0}^n p_{k-h} c_{n-k} \\ &= \sum_{n=0}^{\infty} \gamma_{mn} \sum_{k=0}^{n-h} p_k c_{n-k} \\ &= \lambda_{mn} \end{aligned}$$

Hence

$$\lim_{m \rightarrow \infty} \sum_n \gamma_{mn} s_n = \lim_{m \rightarrow \infty} \gamma_{mn} = 0, \text{ for every fixed } h. \quad (3.11)$$

Thus hypothesis (2.4) is necessary.

(ii) Take $s_n = 1$ in $\{s_n\}$, so that $t_k = 1$, for every k . Then

$$\begin{aligned} \lim_{m \rightarrow \infty} \sum_n \gamma_{mn} s_n &= \lim_{m \rightarrow \infty} \sum_{k=0}^{\infty} P_k \sum_{n=k}^{\infty} \gamma_{mn} c_{n-k} \\ &= \lim_{m \rightarrow \infty} \sum_{n=0}^{\infty} \gamma_{mn} \sum_{k=0}^n p_k c_{n-k} \\ &= \lim_{m \rightarrow \infty} \sum_{n=0}^{\infty} \gamma_{mn} = 0, \end{aligned} \quad (3.12)$$

which is hypothesis (2.5).

(iii) Take $t_k = t + \epsilon_k$, where $\{\epsilon_k\}$ is a null sequence. Then

$$\begin{aligned} \lim_{m \rightarrow \infty} \sum_n \gamma_{mn} s_n &= \lim_{m \rightarrow \infty} t \sum_{k=0}^{\infty} P_k \sum_{n=k}^{\infty} \gamma_{mn} c_{n-k} + \lim_{m \rightarrow \infty} \sum_{k=0}^{\infty} (P_k \sum_{n=k}^{\infty} \gamma_{mn} c_{n-k}) \epsilon_k \\ &= \lim_{m \rightarrow \infty} \sum_{k=0}^{\infty} (P_k \sum_{n=k}^{\infty} \gamma_{mn} c_{n-k}) \epsilon_k, \text{ by (3.12).} \end{aligned}$$

But $\lim_{m \rightarrow \infty} \sum_n \gamma_{mn} s_n = 0$, for all sequences $\{s_n\}$ that are (N, p_n) summable, as such

$$\lim_{m \rightarrow \infty} \sum_{k=0}^{\infty} (p_k \sum_{n=k}^{\infty} \gamma_{mn} c_{n-k}) \epsilon_k = 0,$$

for all null sequences $\{\epsilon_k\}$.

Hence

$$\sup_m \sum_k p_k \left| \sum_{n=k}^{\infty} \gamma_{mn} c_{n-k} \right| \leq M$$

Thus Theorem 1 is established.

4. PROOF OF THEOREM 2.

(If part): We have

$$\begin{aligned} T_m &= \sum_{n=0}^{\infty} a_{mn} s_n \\ &= \sum_{n=0}^{\infty} a_{mn} \sum_{k=0}^n t_k p_k c_{n-k}, \text{ by (3.2),} \\ &= \sum_{k=0}^{\infty} t_k p_k \sum_{n=k}^{\infty} a_{mn} c_{n-k} = \sum_{k=0}^{\infty} b_{mk} t_k, \text{ say,} \end{aligned}$$

where

$$b_{mk} = p_k \sum_{n=k}^{\infty} a_{mn} c_{n-k}. \quad (4.1)$$

In order to establish this part, it is sufficient to show that (b_{mk}) is regular, that is, it belongs to T .

Clearly

$$\sum_k |b_{mk}| = \sum_k p_k \left| \sum_{n=k}^{\infty} a_{mn} c_{n-k} \right| \leq M, \text{ by (2.7).}$$

Since $(a_{mn}) \in T$, for every fixed positive integer n and $\epsilon > 0$,

$$|a_{mn}| < \epsilon, \text{ for all } m > m_0(n, \epsilon), \quad (4.2)$$

and also

$$\sup_{m,n} |a_{mn}| \leq M. \quad (4.3)$$

Now, making use of (4.2) and (4.3), and proceeding along the lines of (3.10), we can easily establish that

$$\lim_{m \rightarrow \infty} b_{mk} = \lim_{m \rightarrow \infty} p_k \sum_{n=k}^{\infty} a_{mn} c_{n-k} = 0, \text{ for every fixed } k.$$

Finally, since

$$\sum_{k=0}^n p_k c_{n-k} = 1, \text{ for every } n,$$

$$\begin{aligned}\sum_{k=0}^{\infty} b_{m k} &= \sum_{k=0}^{\infty} p_k \sum_{n=k}^{\infty} a_{m n} c_{n-k} = \sum_{n=0}^{\infty} a_{m n} \sum_{k=0}^n p_k c_{n-k} \\ &= \sum_{n=0}^{\infty} a_{m n} = A_m \rightarrow 1, \text{ as } m \rightarrow \infty.\end{aligned}$$

Hence $(b_{m k})$ satisfies all the regularity conditions. This proves the "if part" of Theorem 2.

(Only if part): We have

$$\begin{aligned}T_m &= \sum_{n=0}^{\infty} a_{m n} s_n \\ &= \sum_{n=0}^{\infty} a_{m n} \sum_{k=0}^{\infty} t_k p_k c_{n-k} \\ &= \sum_{k=0}^{\infty} t_k p_k \sum_{n=k}^{\infty} a_{m n} c_{n-k} \\ &= \sum_{k=0}^{\infty} b_{m k} t_k, \text{ say.}\end{aligned}$$

Since A includes (N, p_n) , $(b_{m k})$ is regular. Thus

$$\sum_{k=0}^{\infty} |b_{m k}| = \sum_{k=0}^{\infty} p_k \left| \sum_{n=k}^{\infty} a_{m n} c_{n-k} \right| \leq M,$$

which is the required condition.

This completes the proof of Theorem 2.

5. CLOSING REMARKS.

Theorem 1 generalizes the result of Zaman [3]. It assumes a much simplified form if A and B belong to the class T and we have:

THEOREM 3. Let (N, p_n) be a regular Nörlund method and let $\{p_n\} \in M$. Then a necessary and sufficient condition that any two limitation methods, determined by the matrices $A \equiv (a_{m n})$ and $B \equiv (b_{m n})$, belonging to the class T , are equivalent for all such (N, p_n) summable sequences, is that (2.6) hold.

Theorem 2 and 3 together lead to the following Theorem of consistency of matrix limitation methods for (N, p_n) summable sequences.

THEOREM 4. Let (N, p_n) be a regular Nörlund method and let $\{p_n\} \in M$. Then the necessary and sufficient conditions that any two limitation methods, determined by the matrices of the class T , are consistent for all (N, p_n) summable sequences are that they include (N, p_n) .

ACKNOWLEDGEMENT. Authors are indebted to the referee for his valuable criticism and useful suggestions on the original manuscript.

REFERENCES

1. Cooke, R.G. Infinite Matrices and sequence spaces, Macmillan, 1950.
2. Hardy, G.H. Divergent Series, Clarendon Press, Oxford, 1949.
3. Zaman, M.B. On absolute equivalence of T-matrices for (C, r) summable sequences, Jour. Indian Math. Soc. 36 (1972), 89-96.