ON UNIFORM CONVERGENCE FOR (μ, ν)-TYPE RATIONAL APPROXIMANTS IN $\mathbb{C}^n$ - II

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ABSTRACT. This paper shows that if $f(z)$ is analytic in some neighborhood of the origin, but meromorphic in $\mathbb{C}^n$ otherwise, with a denumerable non-accumulating pole sections in $\mathbb{C}^n$, and if for each fixed ν , the pole set of each (μ, ν) - unsolvent rational approximant $\pi_{\mu\nu}(z)$ tends to infinity as $\mu' = \min_{1 \leq i \leq n} (\mu_i) \rightarrow \infty$, then $f(z)$ must be entire in $\mathbb{C}^n$. This paper also shows a monotonicity property for the "error sequence" $e_{\mu\nu} = \|f(z) - \pi_{\mu\nu}(z)\|_K$ on compact subsets K of $\mathbb{C}^n$.

KEY WORDS AND PHRASES. *uniform convergence, entire functions, approximations and expansions.*

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1. INTRODUCTION.

Two earlier papers by Lutterodt [1,2] gave results on uniform convergence under restricted assumptions made about the (μ, ν) - rational approximants. In [1], the B^1 -type (μ, ν) -rational approximants, were assumed to be uniformly bounded on a polydisk; whereas, in [2], the $(\mu, 1)$ -rational approximants were under the assumption that the coefficients of the denominator polynomial of degree $\nu = (1, 1, \dots, 1) = \underline{1}$ vanished as $\mu \rightarrow (\infty, \dots, \infty)$ except for $b_{0 \dots 0}^{(\mu)} \neq 0$. In fact, $b_{0 \dots 0}^{(\mu)}$ is normalized to unity.

In this paper, we attempt to provide a general result about uniform convergence of (μ, ν) -rational approximants to entire functions in $\mathbb{C}^n$.

The main results of this paper are Theorems 1 and 2. Theorem 1 establishes uniform convergence for (μ, ν) unisolvent rational approximants with infinite pole sections that tend to infinity as $\mu \rightarrow (\infty, \dots, \infty)$ on compact subsets of $\mathbb{C}^n$; Theorem 2 introduces an "error sequence"

$$e_{\mu\nu} = \|f(z) - \pi_{\mu\nu}(z)\|_K$$

on any compact-subset of $\mathbb{C}^n$ and shows that $e_{\mu\nu}$ is monotonic in ν for sufficiently large values of μ .

2. NOTATION AND DEFINITIONS.

Let $z = (z_1, \dots, z_n)$ be an n -tuple point in $\mathbb{C}^n$; let $\mu = (\mu_1, \dots, \mu_n)$ and $\nu = (\nu_1, \dots, \nu_n)$ be n -tuples of non-negative integers in $\mathbb{N}^n$.

Let $\mathcal{R}_{\mu\nu}$ be the class of all rational functions of the form

$$R_{\mu\nu}(z) = P_\mu(z)/Q_\nu(z), \quad Q_\nu(0) \neq 0$$

where $P_\mu(z)$ and $Q_\nu(z)$ are polynomials of multiple degree of at most μ and ν , respectively, with $(P_\mu(z), Q_\nu(z)) = 1$ in some neighborhood of the origin.

DEFINITION 1. Suppose $f(z)$ is analytic at the origin and $f(0) \neq 0$. An

$R_{\mu\nu}(z) \in \mathcal{R}_{\mu\nu}$ is said to be a (μ, ν) -type rational approximant to $f(z)$ at $z = 0$ if

$$\frac{\partial |\lambda|}{\partial z^\lambda} (Q_\nu(z)f(z) - P_\mu(z)) \Big|_{z=0} = 0 \quad (2.1)$$

for $\lambda \in E^{\mu\nu} \subset \mathbb{N}^n$, a lattice interpolation set with the following properties:

- (i) $0 \in E^{\mu\nu}$
- (ii) $\lambda \in E^{\mu\nu} \Rightarrow \gamma \in E^{\mu\nu}, \gamma_i \leq \lambda_i \quad i = 1, \dots, n$
- (iii) $E_\mu := \{\lambda \in \mathbb{N}^n: 0 \leq \lambda_i \leq \mu_i, \quad i = 1, \dots, n\} \subset E^{\mu\nu}$
- (iv) $|E^{\mu\nu}| \leq \prod_{i=1}^n (\mu_i + 1) + \prod_{i=1}^n (\nu_i + 1) - 1$
- (v) Each projected variable has the Padé index set
- (vi) Each $\nu_i \leq \mu_i \quad i = 1, \dots, n$.

Here $|E^{\mu\nu}|$ is the cardinality of $E^{\mu\nu}$ and

$$\frac{\partial |\lambda|}{\partial z^\lambda} \equiv \frac{\partial^{\lambda_1 + \dots + \lambda_n}}{\partial z_1^{\lambda_1} \dots \partial z_n^{\lambda_n}}$$

DEFINITION 2. An $R_{\mu\nu}(z) \in \mathcal{R}_{\mu\nu}$ is said to have multiple degree $\mu^* = (\mu_1^*, \dots, \mu_n^*)$ if, in the z_j -variable, $R_{\mu\nu}(z)$ expressed as a quotient of two pseudo-polynomials in z_j , has degree given by $\mu_j^* = \max(\mu_j, \nu_j)$, $1 \leq j \leq n$.

It follows from property (vi) of $E^{\mu\nu}$, that the multiple degree of a (μ, ν) -type rational approximant is always μ .

We shall refer the reader to the definition of a unisolvent (μ, ν) -type rational approximant to $f(z)$ in Lutterodt [3]. We shall denote this by

$$\pi_{\mu\nu}(z) = P_{\mu\nu}(z)/Q_{\mu\nu}(z).$$

We then normalize the denominator polynomial $Q_{\mu\nu}(z)$, dividing numerator and denominator by the modulus of largest coefficient of the denominator polynomial. Thus, we get

$$\pi_{\mu\nu}(z) = P_{\mu\nu}^*(z)/Q_{\mu\nu}^*(z)$$

where $Q_{\mu\nu}^*(z)$ is a normalized polynomial.

3. CONVERGENCE.

The uniform convergence for the (μ, ν) -rational approximants to $f(z)$ entire in $\mathbb{C}^n$ rests on the assumptions made about $f(z)$ and the hypothesis that, for each fixed multiple denominator degree ν of $\pi_{\mu\nu}(z)$, the pole set tends to infinity as $\mu \rightarrow (\infty, \dots, \infty)$. In Theorem 1 below, we assume that $f(z)$ is possibly meromorphic, not with a finite pole set as in Theorem 2 of [3], but with a pole set having infinite sections such that only a finite number of such pole sections overlap with any given polydisk. Thus, Theorem 1 of this paper extends the result in [3].

THEOREM 1: Suppose $f(z)$ is analytic at the origin and is possibly meromorphic with an infinite pole set in $\mathbb{C}^n$ without accumulation of pole sections such that given $\rho > 1$, the polydisk

$$\Delta_\rho^n = \{z \in \mathbb{C}^n: |z_j| < \rho, \quad j = 1, \dots, n, \quad \rho > 1\}$$

overlaps with only a finite number of these pole sections.

Suppose $\pi_{\mu\nu}(z)$ is a unisolvent (μ, ν) -rational approximant to $f(z)$ such that for each fixed ν , the pole set of $\pi_{\mu\nu}(z)$ tends to infinity as $\mu \rightarrow (\infty, \dots, \infty)$. Then

- (i) $f(z)$ must be entire in $\mathbb{C}^n$
- (ii) $\pi_{\mu\nu}(z) \rightarrow f(z)$ uniformly on every compact subset of $\mathbb{C}^n$.

THEOREM 2: Suppose the conditions of Theorem 1 are satisfied. Let K be any compact subset of $\mathbb{C}^n$. Let

$$e_{\mu\nu} = \|f(z) - \pi_{\mu\nu}(z)\|_K = \sup_{z \in K} |f(z) - \pi_{\mu\nu}(z)| \quad (3.1)$$

for each fixed ν .

Then for sufficiently large ν , $e_{\mu\nu}$ is monotonic in ν and satisfies

$$e_{\mu, \nu+1} \leq e_{\mu\nu} \quad \text{with} \quad \nu_j \leq \nu_j + 1, \quad 1 \leq j \leq n.$$

LEMMA 1. Let ν be fixed and let $Q_{\mu\nu}^*(z)$ be a normalized denominator polynomial of $\pi_{\mu\nu}(z)$. The zero set of $Q_{\mu\nu}^*(z)$ tends to infinity as $\mu \rightarrow (\infty, \dots, \infty) = Q_{\mu\nu}^*(z)$ tends to a constant.

PROOF. Suppose the result is false; i.e., for fixed ν , $Q_{\mu\nu}^{-1}(0)$ tends to infinity, but $Q_{\mu\nu}^*(z)$ does not tend to a constant.

By Lemma 1 in [3], given $\rho > 1$ and a polydisk Δ_ρ^n , and μ sufficiently large,

$$Q_{\mu\nu}^{-1}(0) \cap \Delta_\rho^n = \emptyset \quad (3.2)$$

Suppose that $Q_{\mu\nu}^*(z) \rightarrow Q_m^*(z)$ is not constant as $\mu \rightarrow (\infty, \dots, \infty)$ where $m = (m_1, \dots, m_n)$ $m_i \leq \nu_i$, $1 \leq i \leq n$ and that $Q_m^*(z)$ is a polynomial of multiple degree in less than ν in a partial ordered sense. Then since $Q_m^*(z)$ is non-constant, it has a set of non zero coefficients. Thus, $Q_m^{-1}(0)$, the zero set of $Q_m^*(z)$ cannot be empty. Now, taking $\rho_0 > 1$, we find that

$$Q_m^{-1}(0) \cap \Delta_{\rho_0}^n \neq \emptyset \quad (3.3)$$

a contradiction. Hence the above supposition must be false and the Lemma holds.

PROOF OF THEOREM 1. $f(z)$ is analytic at $z = 0$ and is possibly meromorphic with an infinite pole set

$$G = \bigcup_{k=1}^{\infty} G_{\sigma_k}$$

where

$$G_{\sigma_k} \subset G_{\sigma_{k+1}}$$

and

$$G_{\sigma_k} := \{z \in \mathbb{C}^n : q_{\sigma_k}(z) = 0\}.$$

$q_{\sigma_k}(z)$ is a polynomial of at most multiple degree,

$$\sigma_k = (\sigma_{k1}, \dots, \sigma_{kn}).$$

Given any real number $\rho > 1$, and a polydisk Δ_ρ^n , then $\exists k_0 = k_0(\rho)$ such that the zero set $G_{\sigma_{k_0}}$ overlaps the polydisk Δ_ρ^n . Now, by Theorem 1 of [3], if we choose $v = \sigma_{k_0}$, then we must have on Δ_ρ^n as $\mu \rightarrow (\infty, \dots, \infty)$

$$\Delta_\rho^n \cap Q_{\mu v}^{-1}(0) \rightarrow \Delta_\rho^n \cap G_{\sigma_{k_0}} \quad (3.4)$$

But by hypothesis, the pole set of $\pi_{\mu v}(z)$ tends to infinity as $\mu \rightarrow (\infty, \dots, \infty)$ for each fixed v . Therefore, for the given $\rho > 1$ above as $\mu \rightarrow (\infty, \dots, \infty)$, we must have

$$\Delta_\rho^n \cap Q_{\mu v}^{-1}(0) = \emptyset \quad (3.5)$$

Thus by (3.4) and (3.5) we must have

$$\Delta_\rho^n \cap G_{\sigma_{k_0}} = \emptyset.$$

Since $k_0 = k_0(\rho)$ and ρ is arbitrary, it follows that $G_{\sigma_{k_0}}$ must tend to infinity as $k_0 \rightarrow \infty$. Hence, all the poles of $f(z)$ must tend to infinity and $f(z)$ must therefore be entire. This completes (i).

To prove (ii), we note that the result follows immediately from Theorem 1 of [3] and the (i) part just proved above.

PROOF OF THEOREM 2. Let K be any compact subset of $\mathbb{C}^n$. Then we can find $\rho > 1$ and a polydisk $\mathbb{C}^n$ such that $K \subset \Delta_\rho^n$. Then, for μ sufficiently large and $z \in K$, we find by the hypothesis of Theorem 1, that for each fixed v ,

$$Q_{\mu v}^*(z) \neq 0 \quad \text{i.e.} \quad \delta > 0$$

such that

$$|Q_{\mu v}^*(z)| > \delta.$$

Hence, under these conditions, we get

$$\|\pi_{\mu, v+1}(z) - \pi_{\mu v}(z)\|_K \leq \frac{2\|P_{\mu v}^*(z)\|_K}{\delta^2} \|Q_{\mu, v+1}^*(z) - Q_{\mu v}^*(z)\|_K$$

By Lemma 1, we know that $Q_{\mu\nu}^*(z)$ tends to a constant as $\mu \rightarrow (\infty, \dots, \infty)$ for any fixed ν . Hence, given $\varepsilon > 0$, $\mu_0 = (\mu_{10}, \dots, \mu_{n0})$ such that for $\mu_{i0} < \mu_i$, $1 \leq i \leq n$

$$||Q_{\mu, \nu+1}^*(z) - Q_{\mu\nu}^*(z)||_K < \varepsilon \frac{\delta^2}{2M_\rho}. \quad (3.7)$$

$M_\rho = ||P_{\mu\nu}^*(z)||_{\Delta_\rho^n} \geq ||P_{\mu\nu}^*(z)||_K$ by the maximum modulus principle, and M_ρ is dependent on ρ but independent of μ . Hence, by combining (3.6), (3.7) and (3.8) for each fixed ν and $\mu_{i0} < \mu_i$, $1 \leq i \leq n$, we obtain

$$||\pi_{\mu, \nu+1}(z) - \pi_{\mu\nu}(z)||_K < \varepsilon. \quad (3.8)$$

To get the desired inequality, we note by triangular for sup-norms on K that

$$e_{\mu, \nu+1} \leq e_{\mu\nu} + ||\pi_{\mu, \nu+1}(z) - \pi_{\mu\nu}(z)||_K, \quad (3.9)$$

where we have used the definition of $e_{\mu\nu}$ as in (3.1).

For $\mu_{i0} < \mu_i$, $1 \leq i \leq n$, and for each fixed ν ,

$$e_{\mu, \nu+1} < e_{\mu\nu} + \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, the results follows.

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