ON RANK 4 PROJECTIVE PLANES

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ABSTRACT. Let a finite projective plane be called rank m plane if it admits a collineation group G of rank m, let it be called strong rank m plane if moreover \( G_P = G_L \) for some point-line pair \((P,L)\). It is well known that every rank 2 plane is desarguesian (Theorem of Ostrom and Wagner). It is conjectured that the only rank 3 plane is the plane of order 2. By [1] and [7] the only strong rank 3 plane is the plane of order 2. In this paper it is proved that no strong rank 4 plane exists.

KEY WORDS AND PHRASES. Projective planes, rank 4 groups.


1. INTRODUCTION.

In [6] Kallaher gives restrictions for the order \( n \) of a finite rank 3 pro-
jective plane and conjectures that no such plane exists if \( n \neq 2 \). Let a finite projective plane be called a strong rank \( m \) projective plane if it admits a rank \( m \) collineation group \( G \) such that \( G_p = G_1 \) for some point-line pair \((P,1)\). By Bachmann [1] and Kantor [7] no strong rank 3 projective plane of order \( n \neq 2 \) exists. If the conjecture is true that for projective designs the representations on the points and on the blocks of an arbitrary transitive collineation group are similar (see Dembowski [2], p. 78), then every rank \( m \) projective plane is a strong rank \( m \) plane.

We shall prove in this article the following

**THEOREM:** No strong rank 4 projective plane exists.

To prove the Theorem we first divide the strong rank 4 planes into 3 classes (see Lemma 2 and 3). Then we associate with each such plane \((0,1)\)-matrices \( A \) and \( C \) of trace 0 (see [3]). Finally we show that for each class the trace condition contradicts the integrality of the multiplicities of the eigenvalues of \( A \) or \( C \).

We shall use the following notations, definitions and basic results (see Dembowski [2]):

A collineation group of a projective plane has equally many point orbits and line orbits. The rank of a transitive permutation group is the number of orbits of the stabilizer of one of the permuted elements. If \( G \) is a (point or line) transitive collineation group of a projective plane, then the point and line ranks are equal (Kantor [8]). A rank \( m \) projective plane is a projective plane which admits a transitive collineation group whose (point or line) rank is \( m \) (\( m \geq 2 \)). The lines (points) are identified with the set of points (lines) on them. We write \( P \in 1^G \) if and only if \( P \in 1^\gamma \) for all \( \gamma \in G \).
2. **PROOF OF THE THEOREM.**

Let $\mathcal{P} = (P, L, \xi)$ be a projective plane of finite order $n$ and let $G$ be a rank 4 collineation group of $\mathcal{P}$ such that $G_{P_0} = G_{L_0}$ for some point-line pair $(P_0, L_0)$. It is easily seen that $n \geq 3$. A bijective map $\sigma : P \rightarrow L$ is defined by $P^\sigma = 1$ if and only if $P = P_0^\gamma$ and $1 = L_0^\gamma$ for some $\gamma \in G$. If $i \in \mathbb{N}$ we write $l_i$ for $P_i^\sigma$. Clearly $P_0^\sigma = 1_0$ and

$$P_0^\sigma = P_0^\gamma, \quad l_i^{-1} \gamma = l_i^\gamma$$

for all $P \in \mathcal{P}$, $l \in L$, $\gamma \in G$. (1)

For $P \in P$ $G_P$ has exactly 4 orbits $\{P\}$, $\Delta(P)$, $\Gamma(P)$, $\Pi(P)$. We choose the notation in such a way that

$$\Delta(P)^\gamma = \Delta(P^\gamma), \quad \Gamma(P)^\gamma = \Gamma(P^\gamma), \quad \Pi(P)^\gamma = \Pi(P^\gamma)$$

for all $P \in \mathcal{P}$, $\gamma \in G$. (2)

**LEMMA 2.1:** If $A_1, A_2, A_3 \in \{\Delta, \Gamma, \Pi\}$, then $|\Lambda_1(A) \cap \Lambda_2(B)| = |\Lambda_1(A') \cap \Lambda_2(B')|$ if $A \in \Lambda_3(B)$ and $A' \in \Lambda_3(B')$.

**PROOF:** If $A \in \Lambda_3(B)$, $A' \in \Lambda_3(B')$, then for some $\gamma \in G$, $\gamma \in G_B$ $B' = B^\gamma = B_0^\gamma$, $A' = A_0^\gamma$, whence by (2)

$$|\Lambda_1(A') \cap \Lambda_2(B')| = |\Lambda_1(A)^\gamma \cap \Lambda_2(B)^\gamma| = |(\Lambda_1(A) \cap \Lambda_2(B))^\gamma| = |\Lambda_1(A) \cap \Lambda_2(B)| .$$

**LEMMA 2.2:** Suppose that $P_0 \in L_0$. Then $1_0 - \{P_0\}$ and $G_{P_0} - 1_0$ are $G_{P_0}$-orbits, say $\Delta(P_0) = 1_0 - \{P_0\}$ and $G_{P_0} = 1_0 - \{P_0\}$ with $P_2 = \Delta(P_0)$ and $P_3 \in \Pi(P_0)$ can be chosen such that $P_1 \in 1_0, P_0, P_2, P_3 \in 1_2; P_2 \in 1_2; P_1 \notin 1_3$ (Fig. 1).

The case described by Lemma 2 will be called case I.

**PROOF:** If $1_0 - \{P_0\}$ is not a $G_{P_0}$-orbit, then it is the union of 2 orbits, say $1_0 - \{P_0\} = \Delta(P_0) \cup \Gamma(P_0)$. Then $P_0 - \{1_0\}$ is a line orbit $1_0^\sigma$ and $\Pi(P_0) = G_{P_0}$ with $1 = P_0^\sigma$. This leads to the contradiction.
Hence we may assume that \( \Delta(P_o) = 1_o - \{P_o\} \).

Dually: \( P_o - \{1_o\} \) is a \( G_{P_o} \) - orbit, say \( l_2 = P_o - \{1_o\} \) where \( G_{P_o} = \Gamma(P_o) \) (note that \( P_o \notin l_2 \)).

For any point \( Q \in l_2 \), \( P_2 \) on \( l_2 \) holds \( Q = \Pi(P_o) \).

Let \( P'_1 \in \Delta(P_o) \). If \( P_2 \notin l_1 \) put \( P_1 = P'_1 \). If \( P_2 \notin l_1 \) then

\[
|P'_1 G_{P_o} P_2| = |P'_1 G_{P_o} P_2| \geq |l_1 \cap l_2| G_{P_o} P_2| = \frac{n-1}{n} G_{P_o} P_2 |
\]

This implies \( G_{P_o} P_2 = P_1 \) for some point \( P_1 \in \Delta(P_o) \) and hence \( P_2 \in l_1 \).

It remains to prove that \( P_3 \in \Pi(P_o) \cap l_2 \) exists such that \( P_1 \notin l_3 \). If no such \( P_3 \) exists then \( P_1 \notin Q \) for all \( Q \in l_2 \) with \( P_o \) and hence \( G_{P_o} P_2 \leq G_{P_o} P_1 \).

Let \( \gamma \in G \) be such that \( P'_2 = P_o \). Then \( l_2 = l_o \) and therefore \( P_o \in l_0 \), \( P_o \notin Q \) and \( P_o \gamma_0 = P_1 \) for some \( \gamma_0 \in G_{P_o} \). It follows that \( G_{P_o} P_1 = (\gamma_0)^{-1} G_{P_o} P_2 \gamma_0 \).

Hence

\[
G_{P_o} P_1 = G_{P_o} P_2 .
\] (3)

Further

\[
P_2 \notin l_1 \gamma \ \text{for some} \ \gamma \in G_{P_o},
\] (4)

for otherwise \( P'_2 \in l_1 \gamma \) for all \( \gamma \), \( \gamma_0 \in G_{P_o} \) which cannot occur.

\[
P_2 \gamma \in l_1 \ \text{for some} \ \gamma \in G_{P_o} \ \text{if and only if} \ \gamma \in G_{P_2} .
\] (5)

To prove (5) note that by (4) through any point of \( l_2 - \{P_o\} \) goes at least one and hence exactly one line of \( l_1 \gamma \) (3) and \( P_2 \in l_1 \) then imply (5).

Let's apply (5) to \( G_{P_1} \) in place of \( G_{P_o} \):

\[
\Delta(P_1) = l_1 - \{P_2\}; \ \Gamma(P_1) = \Gamma(P_0) \gamma = P_2 \gamma = P_1 .
\]
where $\gamma \in G$ such that $P_0^\gamma = P_1$, $P_2^\gamma = P_3$; $I(P_1) = S_{P_1}$ for some $S \in I_0 - \{P_0, P_1\}$; hence $P_0^\gamma \in I_2$ for some $\gamma \in G_{P_1}$ if and only if $\gamma \in G_{P_0}$.

It follows that $R \notin P_0^\gamma$ for any $R \in I_2 - \{P_0, P_2\}$. Let $R = \gamma$ for some such $\gamma$.

Of the 3 orbits $(P_0, 1_1)^G$, $(P_0, 1_2)^G$, $(P_0, r)^G$ induced by $G$ on $P \times \{1, 2\} - (P_0, 1_0)^G$ only one consists of flags. Thus $(P_1, 1_0)$ and $(P_1, r)$ and then also $(P_0, 1_1)$ and $(R, 1_1)$ belong to the same $G$-orbit. This contradicts $R \notin P_0^\gamma$.

Hence there exists $P_3 \in I(P_0) \cap I_2$ such that $P_1 \notin I_3$.

**LEMMA 2.3:** Suppose that $P_0 \notin I_0$. Then $I_0$ and dually $P_0$ are $G_{P_0}$-orbits, say $\Delta(P_0) = I_0$. $P_1 \in \Delta(P_0)$, $P_2 \in \Gamma(P_0)$, $P_3 \in \Xi(P_0)$ can be chosen such that

- either $P_0, P_2, P_3 \in I_1$; $P_1 \notin I_2, I_3$; $P_2 \notin I_3$; $P_3 \notin I_2$
- or $P_0, P_1, P_3 \in I_2$; $P_2 \notin I_0$; $\Gamma(P_0) \cap I_2 = \{P_2^\gamma\}$ for some $\gamma \in G_{P_0}$; $P_2 \in I_1$; $P_1, P_2, P_3 \notin I_1, I_3$. In both cases $n \geq 4$.

The 2 cases described by Lemma 3 will be called case IIII resp. case II2 (Fig. 2).

**PROOF:** It is easily seen that $I_0$ and $P_0$ are $G_{P_0}$-orbits; say $\Delta(P_0) = I_0$. Let $P_1 \in \Delta(P_0)$. We have to distinguish 2 cases:

- **Case III:** $P_0 \notin I_1$
- **Case II2:** $P_0 \notin I_1$.

**CASE III:** Clearly $P_0 = I_{P_0}$ and $\Gamma(P_0) = I_{P_0}$, $\Xi(P_0) = P_{3, P_0}$ for some $P_2, P_3 \in I_1 - \{P_0, I_0 \cap I_1\}$. If $P_2 \in I_3$ then $(P_2, 1_3) \in (P_0, 1_0)^G$, hence $(P_2, 1_3) \in (P_1, 1_0)^G$, so $P_3 \in I_2$.

![Figure 1](image-url)
Analogously $P_2 \in l_3$ if $P_3 \in l_2$. Thus

$$P_2 \in l_3 \quad \text{if and only if} \quad P_3 \in l_2. \quad (6)$$

Similarly one proves

$$P_1 \in l_2, l_3. \quad (7)$$

If $n > 3$ then, by (6), we can choose $P_2, P_3$ such that $P_2 \notin l_3, P_3 \notin l_2$.

Let's show that $n > 3$ (Fig. 3). Suppose that $n = 3$. Put $P_4 = l_o \cap l_1$.

Then, since $P_o \in l_1$ and $P_1 \in l_o$,

$$l_4 = P_o P_1.$$ 

Let $P_5 \in l_o - \{P_1, P_4\}$.

Then $P_o \in l_5$ and then $l_5 \cap l_2 = (P_o P_5 \cap l_3)P_4 \cap l_2$. Denote this point by $T$. Clearly $P_2P_5 \cap l_2 = T$. Since $(P_2P_5)^{\sigma^{-1}} \in l_2 \cap l_5$ we obtain the contradiction

$$(P_2P_5)^{\sigma^{-1}} \notin P_2P_5.$$
CASE II2: We may assume that $P_0, P_1, P_3 \in \ell_2$ where $P_2 \in \Gamma(P_0)$. Then $G_{P_0}P_1 = G_{P_0}P_2$. We first assume that $n > 3$. Let $1_2 \cap \Gamma(P_0) = \{ \gamma_0 \}$ with some $\gamma_0 \in G_{P_0}$. Then $P_3 \gamma_0 \in 1_2 - \{ P_0, P_1, P_2 \}$ and hence, since $P_2 \gamma_0$ is invariant under $G_{P_0}P_1$ and since $n > 3$, $1_2 \cap 1_2 = P_2 \gamma_0$.

The only $G$-orbit of $\mathfrak{P}_\times \ell$ consisting of flags is $(P_0, 1_2)^G$. Hence $(P_1, 1_2)$, $(P_3, 1_2), (P_1, 1_0) \in (P_0, 1_2)^G$. $P_0 \notin 1_1$ then implies that $(P_2, 1_1), (P_2, 1_3)$, $(P_0, 1_1), (P_2, 1_0) \notin (P_0, 1_2)^G$, in particular $P_2 \notin 1_0, 1_1, 1_3$.

If $P_1 \in 1_3$ then $(P_1, 1_3) \in (P_0, 1_2)^G$ and hence $(P_3, 1_1) \in (P_2, 1_0)^G$.

Since also $(P_0, 1_1) \in (P_2, 1_0)^G$ we have $P_0 \gamma_1 = P_3$ for some $\gamma_1 \in G_{1_1} = G_{P_1}$. This implies that $G_{P_1}$ is transitive on $1_2 - \{ P_1, P_2 \}$ which is impossible. Hence $P_1 \notin 1_3$.

If $n = 3$ then $1_2 = \{ P_0, P_1, P_2, P_3 \}$. $\gamma_0$ is of order 4, for if $\gamma_0^2 = 1$ then $(P_2, 1_2) \in (P_2, 1_2)$ which is impossible. Moreover $P_2 \neq P_0$ since otherwise $\gamma_0^2 \in G_{P_0}, P_2 = 1$. It follows that $|G_{P_0}P_2| = 4$ which contradicts $G_{P_0}P_2 = \{ P_0 \gamma_0^2 \}, P_2 \gamma_0^2 \}$. This completes the proof of the Lemma.

Let us now associate with $(G, \mathfrak{P})$ 3 $(0,1)$-matrices.

If $P \in \mathfrak{P}$ is a $G$-orbit then let $P'(P)$ denote the paired orbit (see Wielandt [9]). If $Q \in \mathfrak{P}(P)$ then $Q = P'$ for some $\gamma \in G$ and $Q' \in (\mathfrak{P}(P))' = \mathfrak{P}(P') = \mathfrak{P}(Q)$.

Hence $Q' = P \in P'(Q)$, i.e.

$Q \in \mathfrak{P}(P)$ implies that $P \in \mathfrak{P}'(Q)$.  

(8)
This implies that in

Case I:  Case III:  Case II2:

\[ \Delta'(P) = \Gamma(P) \quad \Delta'(P) = \Delta(P) \quad \Delta'(P) = \Gamma(P) \]
\[ \Gamma'(P) = \Delta(P) \quad \Gamma'(P) = \Pi(P) \text{ resp. } \Gamma(P) \quad \Gamma'(P) = \Delta(P) \]
\[ \Pi'(P) = \Pi(P) \quad \Pi'(P) = \Gamma(P) \text{ resp. } \Pi(P) \quad \Pi'(P) = \Pi(P). \]

Now let \( P = \{P_1, P_2, \ldots, P_v\} \), \( L = \{1, 2, \ldots, v\} \), \( \lambda_k = P_k^o \) (k = 1, 2, \ldots, v). Let \( A \) be the \((0,1)\)-matrix with rows enumerated by the points \( P_k \) and columns by \( \Delta(P_k) \) and such that \( (P_k, \Delta(P_k)) = 1 \) if and only if \( P_k \in \Delta(P_k) \). Let \( B, C \) be the analogous matrices with \( \Gamma(P_k) \) resp. \( \Pi(P_k) \) in place of \( \Delta(P_k) \).

We have in

**Case I:**

\[ A^t = B, \quad C^t = C \]

**Case III:**

\[ A^t = A, \quad B^t = C \quad \text{if } \Gamma'(P) = \Pi(P) \quad A^t = B, \quad C^t = C \]

**Case II2:**

\[ A^t = A, \quad B^t = B, \quad C^t = C \quad \text{if } \Gamma'(P) = \Gamma(P) \]

Let \( k = \vert \Delta(P) \vert \), \( l = \vert \Gamma(P) \vert \), \( m = \vert \Pi(P) \vert \),

\[ \vert \Delta(P) \cap \Delta(Q) \vert = \begin{cases} \lambda \quad \text{if } Q \in \Delta(P) \\ \mu \quad \text{if } Q \in \Pi(P) \end{cases} \]

\[ \vert \Pi(P) \cap \Pi(Q) \vert = \begin{cases} \lambda' \quad \text{if } Q \in \Delta(P) \\ \mu' \quad \text{if } Q \in \Gamma(P) \end{cases} \]

A straightforward calculation shows that

\[ I + A + B + C = J, \quad \text{the } v \times v \text{-matrix with 1's in every entry} \]

\[ A^t A = k I + \lambda A + \mu B + \nu C \]

\[ C^t C = m I + \mu'A + \nu'B + \lambda'C \]
Now we determine the eigenvalues of $A$ in case III and of $C$ in the cases I and II2.

**CASE III:** \( k = n + 1 \)

\[
l = n_2(n + 1) \quad \text{where} \quad n_2 = |P_2^{G_{P_0,P_1}}|\]

\[
m = n_3(n + 1) \quad \text{where} \quad n_3 = |P_3^{G_{P_0,P_1}}|\]

\[
k + 1 + m + 1 = v = n^2 + n + 1, \quad n_2 + n_3 = n - 1, \quad \lambda = \mu = \nu = 1.
\]

It follows that \( A^2 = A^tA = (n + 1)I + A + B + C = nI + J \); hence

\[
(A - (n + 1)I)(A^2 - nI) = 0. \quad \text{This gives the eigenvalues of } A:
\]

\[
\lambda_1 = n + 1, \quad \lambda_{2,3} = \pm \sqrt{n}.
\]

**CASE I:** \( k = 1 = n, \ m = n(n - 1), \ k + 1 + m + 1 = v = n^2 + n + 1. \)

We have

\[
\lambda' = |\Pi(P_o) \cap \Pi(P_3)|
\]

\[
\mu' = |\Pi(P_o) \cap \Pi(P_1)|
\]

\[
\nu' = |\Pi(P_o) \cap \Pi(P_2)|.
\]

Let's calculate \( \lambda' \):

\[
n(n - 1) = |\Pi(P_3)| = |\Pi(P_3) \cap \Delta(P_o)| + |\Pi(P_3) \cap \Gamma(P_o)| + |\Pi(P_3) \cap \Pi(P_o)| + 1 \quad (9)
\]

(note that \( \Gamma(P_3) = P_2^{G_{P_0,P_3}} \) and hence \( P_o \in \Pi(P_3) \)).

\[
n = |\Delta(P_o)| = |\Delta(P_o) \cap \Delta(P_3)| + |\Delta(P_o) \cap \Gamma(P_3)| + |\Delta(P_o) \cap \Pi(P_3)|. \quad (10)
\]

Clearly

\[
|\Delta(P_o) \cap \Delta(P_3)| = 1 \quad (11)
\]

\[
|\Delta(P_o) \cap \Gamma(P_3)| = |\Delta(P_3) \cap \Gamma(P_o)| = 2. \quad (12)
\]
PROOF of (12): \( P_1 \in \pi(P_3) \) and \( P_3 \in \pi(P_1) \), hence, by Lemma 1, \( |\Delta(P_1) \cap \Gamma(P_3)| = |\Delta(P_3) \cap \Gamma(P_1)| \) = \( |\Delta(P_3) \cap \Gamma(P_1)| \) for some \( \gamma_o \in G_{P_1} \). Thus \( |P_1 \cap P_2| = n - 1 \) implies that \( |1_{P_1} \cap G_{P_1} \cap P_2| > n - 1 \). Hence \( |1_{P_1} \cap G_{P_1} \cap P_2| = n - 1 \). Since \( P_1 \not\subset P_2 \) we then have \( P_1 \cap P_2 = P_1 \), i.e. \( G_{P_1} \cap P_2 \). Since both groups are conjugate (see the proof of Lemma 2) this gives \( G_{P_1} \cap P_2 = G_{P_2} \). Thus \( P_1 \not\subset P_2 \) for some \( \gamma_o \) in \( G_{P_1} \). Since, by the above, \( G_{P_1} \cap P_2 \) is transitive on \( \{P_1, P_2\} \), \( |P_1 \cap P_2| = n - 1 \); hence \( |1_{P_1} \cap P_2| = |1_{P_1} \cap P_2| = 2 \). This proves (12).

Equations (10), (11), (12) imply
\[
|\pi(P_3) \cap \Delta(P_1)| = n - 3. \tag{13}
\]
To determine \( |\pi(P_3) \cap \Gamma(P_1)| \) we use
\[
n = |\Gamma(P_1)| = |\Gamma(P_1) \cap \Delta(P_3)| + |\Gamma(P_1) \cap \Gamma(P_3)| + |\Gamma(P_1) \cap \pi(P_3)|. \tag{14}
\]
By (12) \( |\Gamma(P_3) \cap \Delta(P_3)| = 2 \). Since \( \Gamma(P_3) = G_{P_2} \), \( |\Gamma(P_1) \cap \Gamma(P_3)| = G_{P_2} \cap G_{P_3} = |1_{P_1} \cap 1_{P_2}| = 1 \). It follows that
\[
|\pi(P_3) \cap \Gamma(P_1)| = n - 3. \tag{15}
\]
Equations (9), (13) and (15) imply that
\[
\lambda' = n^2 - 3n + 5. \tag{16}
\]
Analogously we calculate \( \mu' \) and \( \nu' \):
\[
n(n - 1) = |\pi(P_1)| = |\pi(P_1) \cap \Delta(P_1)| + |\pi(P_1) \cap \Gamma(P_1)| + |\pi(P_1) \cap \pi(P_1)| \]
with \( |\pi(P_1) \cap \Delta(P_1)| = n - 1 \).
In
\[
n = |\Gamma(P_1)| = |\Gamma(P_1) \cap \Delta(P_1)| + |\Gamma(P_1) \cap \Gamma(P_1)| + |\Gamma(P_1) \cap \pi(P_1)|
\]
\[ |\Gamma(P_o) \cap \Delta(P_1)| = 1 \] by the proof of (12) and \[ |\Gamma(P_o) \cap \Gamma(P_1)| = G_{P_1} \cap G_{P_0} = |1_2^{P_0} \cap 1_2^{P_1}| = 0. \] Hence \[ |\Pi(P_1) \cap \Gamma(P_o)| = n - 1 \] and thus
\[ n' = (n - 1)(n - 2). \] (17)

\[ n(n - 1) = |\Pi(P_2)| = |\Pi(P_2) \cap \Delta(P_o)| + |\Pi(P_2) \cap \Gamma(P_o)| + |\Pi(P_2) \cap \Pi(P_o)|. \]

In
\[ n = |\Delta(P_o)| = |\Delta(P_o) \cap \Delta(P_2)| + |\Delta(P_o) \cap \Gamma(P_2)| + |\Delta(P_o) \cap \Pi(P_2)| \]
\[ |\Delta(P_o) \cap \Delta(P_2)| = 0 \] and \[ |\Delta(P_o) \cap \Gamma(P_2)| = |P_1^{P_0} \cap P_1^{P_2}| = |1_1^{P_0} \cap 1_1^{P_2}| = 1 \]
(note that \[ |1_1^{P_2}| = |P_1^{P_2}| = n \] and hence \( \Gamma(P_2) = P_1^{P_2}. \) Hence \[ |\Pi(P_2) \cap \Delta(P_o)| = n - 1. \]

Further \[ |\Pi(P_2) \cap \Gamma(P_o)| = |\Gamma(P_o)| - |\Gamma(P_2) \cap \Delta(P_o)| - |\Gamma(P_o) \cap \Gamma(P_2)| - 1 \]
where \[ |\Gamma(P_o)| = n, |\Gamma(P_o) \cap \Delta(P_2)| = 0 \] and \[ |\Gamma(P_o) \cap \Gamma(P_2)| = |P_2^{P_0} \cap P_1^{P_2}| = G_{P_1} \cap G_{P_2} = 0. \] Hence \[ |\Pi(P_2) \cap \Gamma(P_o)| = n - 1. \] It follows that
\[ n' = (n - 1)(n - 2). \] (18)

Equations (16), (17) and (18) imply that \[ c^2 = C^3 = n(n - 1) I + (n - 1)(n - 2)(A + B) + (n - 3n + 5) C = n(n - 1) I + (n - 1)(n - 2)(J - I) + 3C \]
and then \( (C - n(n - 1) I)(C^2 - 3C - 2(n - 1) I) = 0. \)

The eigenvalues of \( C \) are \[ \lambda_1 = n(n - 1); \lambda_{2,3} = (3 \pm \sqrt{8n + 1})/2. \]

CASE II2: \( k = 1 = n + 1, m = (n - 2)(n + 1), k + 1 + m + 1 = n = n^2 + n + 1. \)

By the proof of Lemma 3 \( n \geq 4. \) Let's determine \( \lambda', \mu', \nu': \)

\[ (n + 1)(n - 2) = |\Xi(P_3)| = |\Xi(P_3) \cap \Delta(P_o)| + |\Xi(P_3) \cap \Gamma(P_o)| + |\Xi(P_3) \cap \Pi(P_o)| - 1. \]
In \( n + 1 = |\Delta(P_o)| = |\Delta(P_o) \cap \Delta(P_2)| + |\Delta(P_o) \cap \Gamma(P_2)| + |\Delta(P_o) \cap \Pi(P_2)| \)
\( |\Xi(P_3)| \) clearly \[ |\Delta(P_o) \cap \Delta(P_3)| = 1. \] Let's show that
\[ |\Delta(P_o) \cap \Gamma(P_2)| = 2 \] (19)
\[ |\Delta(P_1) \cap \Gamma(P_o)| = 2. \] (20)
PROOF of (19) and (20): By Lemma 1 \(|\Delta(P_0) \cap \Gamma(P_3)| = |\Delta(P_3) \cap \Gamma(P_0)|\).

For \(\gamma'_o \in G_{P_0}\)
\[
\gamma_o', \quad \gamma_o' \in G_{P_0}, \quad \gamma_o' \neq P_2 \quad \text{if and only if} \quad \gamma_o' \in G_{P_0}, P_1
\]
for otherwise \(\gamma_o' \in \gamma_o P_{\gamma_o' P_1}, \quad \gamma_o' \in \gamma_o G_{P_0} P_1, \quad |\gamma_o G_{P_0} P_1| = n - 2, \quad P_2 \in \gamma_o G_{P_0} P_1, \quad \gamma_o' P_1, \quad l_1 \gamma_o P_{\gamma_o' P_1}, \quad l_1 \gamma_o G_{P_0} P_1, \quad l_1 \gamma_o G_{P_0} P_1\), which leads to the contradiction \(n + 1 = |l_1 \gamma_o| > (|P_2| - 2) + (|P_2| - 2) = 2(n - 1)\).

Further
\[
\gamma_o' \notin l_1 \cup P_2 P_2 \quad \text{for some} \quad \gamma_o \in G_{P_0}.
\]
otherwise, since \(|P_2 P_2| > 5\), every line of \(l_1 P_2\) would contain at least 3 points of \(P_2 P_2\) and this would imply that a point of \(P_2 P_2 - (l_1 \cup P_2 P_2)\) exists. By (21) and (22) \(\gamma_o' \neq P_2 \quad \text{for some} \quad \gamma_o \in G_{P_0} P_0, P_1\) for some \(\gamma_o \notin G_{P_0}\) for some \(\gamma_o \in G_{P_0}\). Since \(|l_3 P_0 P_1| = n - 2, \quad |P_2 P_0 l_3| = |\Gamma(P_0) \cap \Delta(P_3)| = 2. \quad \text{This proves (19).}

Each of the \(n - 2\) lines of \(l_3 P_0 - \{\gamma_o P_2\}\) through \(\gamma_o P_2\) contains exactly one point of \(P_2 P_2 - \{\gamma_o P_2\}\). Together with \(P_2 P_2\), this gives \(n\) points of \(G_{P_0}\). It follows that exactly one point of \(P_2 P_2 - \{\gamma_o P_2\}\) lies on \(l_1 P_2\). This proves (20).

By means of (19) we obtain \(|\Pi(P_3) \cap \Delta(P_0)| = n - 2\).

In \(n + 1 = |\Gamma(P_0)| = |\Gamma(P_0) \cap \Delta(P_3)| + |\Gamma(P_0) \cap \Gamma(P_3)| + |\Gamma(P_0) \cap \Pi(P_3)|\)
\(|\Gamma(P_0) \cap \Delta(P_3)| = 2 \quad \text{by (19) and} \quad |\Gamma(P_0) \cap \Gamma(P_3)| = |P_2 P_0 \cap P_2 P_3| = |l_2 P_0 \cap l_2 P_3| = 1. \quad \text{Hence} \quad |\Pi(P_3) \cap \Gamma(P_0)| = n - 2 - 3n + 1.

\(\mu' = |\Pi(P_0) \cap \Pi(P_1)| = |\Pi(P_1)| - |\Pi(P_1) \cap \Delta(P_0)| - |\Pi(P_1) \cap \Gamma(P_0)| \quad \text{where} \quad |\Pi(P_1)| = (n + 1)(n - 2), \quad |\Pi(P_1) \cap \Delta(P_0)| = n - 2 \quad \text{and} \quad |\Pi(P_1) \cap \Gamma(P_0)| = |\Gamma(P_0)| - |\Gamma(P_0) \cap \Delta(P_1)| - |\Gamma(P_0) \cap \Gamma(P_1)| - |\Gamma(P_0) \cap \Delta(P_1)| - 2 \quad \text{by (20) and} \quad |\Gamma(P_0) \cap \Gamma(P_1)| = |P_2 P_0 \cap P_2 P_1| = |l_2 P_0 \cap l_2 P_1| = 1. \quad \text{Hence} \quad |\Pi(P_1) \cap \Gamma(P_0)| = n - 2 \quad \text{and} \quad \mu' = \ldots
\[(n - 2)(n - 1).\]

\[\nu' = |\Pi(P_o) \cap \Pi(P_2)| = |\Pi(P_2)| - |\Pi(P_2) \cap \Delta(P_o)| - |\Pi(P_2) \cap \Gamma(P_o)|\]

where

\[|\Pi(P_2)| = (n + 1)(n - 2),\]

\[|\Pi(P_2) \cap \Delta(P_o)| = |\Pi(P_o) \cap \Delta(P_1)|\] by Lemma 1

\[= |\Delta(P_1)| - |\Delta(P_1) \cap \Delta(P_o)| - |\Delta(P_1) \cap \Gamma(P_o)|\]

\[= (n + 1) - 1 - 2 = n - 2\] by (20),

\[|\Pi(P_2) \cap \Gamma(P_o)| = |\Pi(P_o) \cap \Gamma(P_1)|\] by Lemma 1

\[= |P_3 \cap P_1| = |\{\gamma_0', \gamma_0 \in G, P_1 \in \gamma_0\}| = n - 2\] since through any point on \(1_o\) goes exactly one line of \(1_1\) and one of \(1_2\). Hence

\[\nu' = (n - 2)(n - 1).\]

It follows that

\[C^2 = C^t C = (n + 1)(n - 2)I + (n - 1)(n - 2)(A + B) + (n^2 - 3n + 1)C = (n + 1)(n - 2)I + (n^2 - 3n + 2)(A + B + C) - C\]

and \(C^2 + C - 2(n - 2)I = (n + 1)(n - 2)J\) whence

\[(C - (n + 1)(n - 2)I)(C^2 + C - 2(n - 2)I) = 0.\]

The eigenvalues of \(C\) are \(\lambda_1 = (n + 1)(n - 2), \lambda_2, \lambda_3 = (-1 + \sqrt{8n - 15'})/2.\)

REMARK: Let \(\phi: G \rightarrow \text{GL}_V(\mathbb{C})\) be the matrix representation of \(G\) obtained by associating with each \(\gamma \in G\) the corresponding permutation matrix \(\phi(\gamma)\) (the ordering of \(P\) is the same as used in constructing the matrices \(A, B, C\)). By (2) \(\phi(\gamma)\) commutes with \(A, B, C\) for all \(\gamma \in G\). Hence, by [9] Theorem 28.4, \(\{I, A, B, C\}\) is the basis of the commuting algebra \(W(G)\) of \(\phi\). By [9] Theorem 29.5 \(W(G)\) is commutative and hence, by [9] Theorem 29.4, the representation \(\phi\) has 4 irreducible constituents \(D_1, D_2, D_3, D_4\), each with multiplicity 1. If \(f_i\) is the degree of \(D_i\) then \(f_1 = 1\) and \(\sum_{i=1}^{4} f_i = \nu\).

Let us finally show how the fact that \(A\) and \(C\) have trace 0 contradicts the integrality of the multiplicities of \(\lambda_1, \lambda_2, \lambda_3\).
In the 3 cases $\lambda_1$ appears with multiplicity 1. Let $f$ denote the multiplicity of $\lambda_2$; then $v - f - 1$ is the multiplicity of $\lambda_3$. This leads to

$0 = n(n - 1) + f(3 + \sqrt[8n+1])/2 + (n(n + 1) - f)(3 - \sqrt[8n+1])/2$ in case I,

$0 = (n + 1) + f\sqrt{n} + (-\sqrt{n})(n(n + 1) - f)$ in case III,

$0 = (n + 1)(n - 2) + f(-1 + \sqrt[8n-15])/2 + (n(n + 1) - f)(-1 - \sqrt[8n-15])/2$ in case I1.

In any case this contradicts the fact that $n > 2$ and $f > 1$ are integers.

In case I1 this is clear.

In case I suppose that a prime $p$ divides $\sqrt[8n+1]$. Then $p \nmid n$, hence $p \mid 5n + 1$ and then $p \mid 3n$, i.e. $p = 3$. This implies that $8n + 1 = 3^{2i}$ for some $i > 2$ and that $n(5n + 1)/\sqrt[8n+1] = (3^{2i} - 1)(5 \cdot 3^{2i-1} + 1)/8^{2i} \notin \mathbb{N}$.

In case I12 suppose that a prime $p$ divides $\sqrt[8n-15]$. Then $p \in \{17, 23\}$ and $p^2 \nmid n + 1, p^2 \nmid n - 4$. Hence $8n - 15 \in \{17^2, 23^2, 17^2 \cdot 23^2\}$, i.e. $n \in \{38, 68, 19112\}$. Suppose that $n = 38$. Since $G_{P_0, P_2}$ is transitive on $1^2 - \{P_0, P_1, P_2\}$, $|G|$ is even. This contradicts the fact that if $n \equiv 2 \pmod{4}$, then the full collineation group is of odd order (Hughes [5]).

Suppose that $n \in \{68, 19112\}$. Then $n$ is not a square and $n^2 + n + 1$ not a prime. Hence, since $G$ is flag-transitive, $n$ is a prime power (Higman and Mc Laughlin [4]) which is absurd.

REFERENCES


