ABSTRACT. It is proved here that a completely regular Hausdorff space $X$ is pseudocompact if and only if for any continuous function $f$ from $X$ to a pseudocompact space (or a compact space) $Y$, $f^*\phi$ is z-ultrafilter whenever $\phi$ is a z-ultrafilter on $X$.

KEY WORDS AND PHRASES. Pseudocompact, $\beta X$, z-filter, z-ultra function.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. Primary 54D99.

1. INTRODUCTION.

For notations and basic results one is referred to [1]. We only consider here completely regular Hausdorff spaces.

Let $f$ be continuous from $X$ to $Y$. Let $\phi$ be a z-ultrafilter on $X$, then $f^*\phi$ denotes the z-filter $\{B \in Z(Y): f^{-1}(B) \in \phi\}$ on $Y$ and is known to be prime. We further know that a prime z-filter is contained in a unique z-ultrafilter. Let $\Delta(f)\phi$ denote the z-ultrafilter containing $f^*\phi$. Thus we have a function $\Delta(f)$ from $\beta X$ to $\beta Y$ sending $\phi$ to $\Delta(f)\phi$. The function $f$ is called z-ultra if $f^*\phi = \Delta(f)\phi$ for every z-ultrafilter $\phi$ on $X$. 
2. MAIN RESULTS

**PROPOSITION.** A continuous function \( f \) from \( X \) to \( Y \) is \( z \)-ultra if and only if for every zero-set \( B \) in \( Y \), \( \Delta(f)^{-1}(\beta^Y) = f^{-1}(B) \).

**PROOF.** Let \( f \) be \( z \)-ultra. Then, \( \phi \in \Delta(f)^{-1}(\beta^Y) \) if and only if \( \Delta(f)\phi = f^*\phi \in \beta^Y \). But this is equivalent to \( B \in f^*\phi \) or to \( f^{-1}(B) \in \phi \), which happens if and only if \( \phi \in f^{-1}(B) \).

Conversely, \( B \in f^*\phi \) and only if \( \phi \in f^{-1}(B) \), i.e. \( \Delta(f)\phi \in \beta^Y \), since \( f^{-1}(B) = \Delta(f)^{-1}(B) \). But \( \Delta(f)\phi \in \beta^Y \) is equivalent to saying that \( B \in \Delta(f)\phi \).

We see that \( f^* = \Delta(f) \).

In order to prove the main theorem of the paper we need the following observations for pseudocompact spaces. If \( X \) is pseudocompact, then a subset of \( \beta X \) is a zero-set if and only if it is closure of a zero-set in \( X \) and conversely, a subset of \( X \) is a zero-set in \( X \) if and only if its closure is so in \( \beta X \).

**THEOREM.** If a space \( X \) is pseudocompact then any continuous function \( f \) from \( X \) to any pseudocompact space \( Y \) is \( z \)-ultra. Conversely, if the inclusion of \( X \) in \( \beta X \) is \( z \)-ultra, then \( X \) is pseudocompact.

**PROOF.** Let \( B \) be a zero-set in \( Y \). Since \( \beta^Y \) is a zero-set in \( \beta Y \) as \( Y \) is pseudocompact, \( \Delta(f)^{-1}(\beta^Y) \) is a zero-set in \( \beta X \). Pseudocompactness of \( X \) implies that \( \Delta(f)^{-1}(\beta^Y) = \beta^X \) for some zero-set \( A \) in \( X \). We show that \( A = f^{-1}(B) \).

Since \( \Delta(f)/X = f \), we observe that \( \Delta(f)^{-1}(B) \cap X = f^{-1}(B) \). Clearly, \( \Delta(f)^{-1}(\beta^Y) \cap X = \Delta(f)^{-1}(B) \cap X = f^{-1}(B) \). Next, \( \Delta(f)^{-1}(\beta^Y) \cap X = \beta^X \cap X = A \). Hence \( f^{-1}(B) = A \), and we have \( f \) to be \( z \)-ultra.

Conversely, let \( i \) be the inclusion of \( X \) in \( \beta X \). Since \( \Delta(i)/X = i \), \( \Delta(i) \) is the identity on \( \beta X \). Let \( B \) be a nonempty zero-set in \( \beta X \). Since \( i \) is \( z \)-ultra, from the above proposition we have that \( B = \Delta(i)^{-1}(B) = i^{-1}(B) \). But \( i^{-1}(B) \cap X = \beta^X \cap X = \beta^X \cap X = \beta^X \cap X = \beta^X \cap X = A \). Hence \( f^{-1}(B) = A \), and we have \( f \) to be \( z \)-ultra.

As an application of our theorem we prove the following well known theorem due to Glicksberg [2].
THEOREM. If $X$ is pseudocompact and $Y$ is compact, then $X \times Y$ is pseudocompact.

PROOF. Let $f: X \times Y \to Z$ be a continuous function, $Z$ some pseudocompact space. Consider a $z$-ultrafilter $\phi$ on $X \times Y$. Let $\pi_2: X \times Y \to Y$ denote the projection on the second coordinate. Since $Y$ is compact and $\pi_2^*\phi$ is a $z$-filter, it is fixed as well. Let $y_0 \in \bigcap \pi_2^*\phi$. Hence $\phi_1$, the restriction of $\phi$ to the subspace $X \times \{y_0\}$ is a $z$-ultrafilter on $X \times \{y_0\}$. Let $f_1$ denote the restriction of $f$ to the subspace $X \times \{y_0\}$. Since $X$ is pseudocompact, $f_1$ is $z$-ultra. Clearly, $f_1^* \subseteq f_1^*\phi$. Next, let $B \in f_1^*\phi$. Hence $f_1^{-1}(B) \in \phi_1$. Since $f_1^{-1}(B)$ contains $f_1^{-1}(B)$, $f_1^{-1}(B)$ intersects every member of $\phi$. Thus $f_1^{-1}(B) \in \phi$ as it is a $z$-ultrafilter. We get that $B \in f^*\phi$. Hence $f^*\phi = f_1^*\phi_1$ and it follows that $f$ is $z$-ultra.

ACKNOWLEDGEMENT

This work was done while the first author was visiting Mehta Research Institute, Allahabad in summer 1977.

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