THE STAR COMPACTIFICATION

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ABSTRACT. The relationships between a convergence space and its star compactification is studied. Special attention is given to lifting properties of this compactification. In particular, it is shown that a natural extension of any continuous function to the respective compactification spaces is θ -continuous.

KEY WORDS AND PHRASES. *Convergence space, compactification, G-space, R-series, natural extension, θ -continuous function, proper map, open map.*

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1. INTRODUCTION.

We study a convergence space compactification which was introduced by one of the authors in 1970 (see [11]). The star compactification $\kappa^* = (X^*, i^*)$ of a convergence space X is constructed by adjoining to X the set X' of all non-convergent ultrafilters on X and constructing a compact convergence structure on $X^* = X \cup X'$ in a natural way. It is proved in [11] that a continuous function from a T_2 space X into a compact T_3 space Y has a continuous extension to X^* .

The authors published a survey paper, [7], concerning the existence of largest and smallest convergence space compactifications relative to various constraints. In all cases studied, the largest compactification, whenever it existed, turned out to be κ^* . In a more recent paper, [9], we showed that κ^* can be used to characterize ω -regular and completely regular spaces. These results suggest that further investigation of the star compactification is appropriate.

In Section 2, we examine the relationship between the decomposition series of X and X^* , showing that the lengths of these series can differ by at most one. These results yield a method for constructing compact T_2 spaces with arbitrarily long decomposition series. In Section 3, the R -series of X and X^* are compared. By means of the R -series, the notion of θ -continuity and other θ -mapping properties (see [2], [3]) are extended to convergence spaces.

If f is a function from a space X to a space Y , then a "natural extension" $f_* : X^* \rightarrow Y^*$ is defined in Section 4. The natural extension is unique if Y is T_2 and coincides with the continuous extension constructed in [11] when Y is compact and T_3 . The main result of Section 4 is that any natural extension f_* is θ -continuous whenever f is continuous. This result is used to obtain, among other things, an alternate construction of βX for a Tychonoff topological space X . Section 5 examines conditions on f , X , and Y under which f_* is continuous, and Section 6 gives conditions under which f_* preserves certain quotient-type mapping

properties, such as "open", "proper", and "perfect".

2. DECOMPOSITION SERIES.

The reader is asked to refer to [7] for convergence space notation and terminology not given here, as well as additional information about the star compactification. As in [7], space will always mean convergence space, and the abbreviation "u.f." is often used for "ultrafilter". The separation axioms T_1 (singletons are closed), T_2 (convergent filters have unique limits), and T_3 (regular plus T_2) will be used, but no separation axioms are assumed unless such is explicitly stated.

Given a space X , let $F(X)$ (resp. $U(X)$) be the set of all filters (resp., ultrafilters) on X . Let X' be the set of all non-convergent members of $\mathcal{U}(X)$, and $X^* = X \cup X'$. If $A \subseteq X$, define $A' = \{ \mathcal{F} \in X' : A \in \mathcal{F} \}$, and $A^* = A \cup A'$. If $\mathcal{F} \in F(X)$, and $F' \neq \emptyset$ for all $F \in \mathcal{F}$, then let $\mathcal{F}'$ be the filter generated by $\{F' : F \in \mathcal{F}\}$; let $\mathcal{F}^*$ be the filter generated by $\{F^* : F \in \mathcal{F}\}$. If $\mathcal{F}'$ exists, then $\mathcal{F}^* = \mathcal{F} \cap \mathcal{F}'$; otherwise, $\mathcal{F}^* = \mathcal{F}$. We omit the easy proofs of the first two propositions.

PROPOSITION 2.1. The following equalities hold for any subsets A, B of X :
 $A' \cup B' = (A \cup B)'$; $A' \cap B' = (A \cap B)'$; $A^* \cup B^* = (A \cup B)^*$; $A^* \cap B^* = (A \cap B)^*$.

Let X be a space, $G \in F(X^*)$, and $X' \in G$. Define $G^\wedge$ to be the filter on X generated by $\{A \subseteq X : A' \in G\}$.

PROPOSITION 2.2. (a) If $\mathcal{F} \in F(X)$, $\mathcal{L} \in F(X^*)$, and $\mathcal{L} \geq \mathcal{F}^*$, then $\mathcal{L}^\wedge \geq \mathcal{F}$.

(b) If $\mathcal{F} \in \mathcal{U}(X)$ and $\mathcal{F}'$ exists, then $(\mathcal{F}')^\wedge = \mathcal{F}$.

(c) If $G \in \mathcal{U}(X^*)$ and $X' \in G$, then $G^\wedge \in \mathcal{U}(X)$.

A convergence structure is defined on X^* as follows:

For $x \in X$, $G \rightarrow x$ in X^* iff there is $\mathcal{F} \rightarrow x$ in X such that $G \geq \mathcal{F}^*$; for $\mathcal{L} \in X'$, $G \rightarrow \mathcal{L}$ iff $G \geq \mathcal{L}^*$. Let i^* denote the identity embedding of X into X^* ; it is proved in [11] that $\kappa^* = (X^*, i^*)$ is a compactification of X which is T_2

whenever X is T_2 . It is immediate from the construction that, for any non-compact space X , X^*-X is a T_2 pretopological space; thus X^* is pretopological whenever X is pretopological. The universal property of κ^* established in [11] will be obtained in Section 3 as a corollary of a much more general result.

A subset A of space X is bounded if each ultrafilter containing A is convergent. X is said to be locally bounded if each convergent filter contains a bounded set. X is essentially bounded if $\mathfrak{F} \in X'$ implies that the filters

$\mathfrak{F}$ and $\cap \{ \mathfrak{G} \in X' : \mathfrak{G} \neq \mathfrak{F} \}$ contain disjoint sets. The next proposition is proved in [7].

PROPOSITION 2.3. (a) X is locally bounded iff X is open in X^* .

(b) X is essentially bounded iff $X^* - X$ is discrete.

We shall next consider the relationship between the closure operators of X and X^* . Let cl_X be the closure operator on a space X . For an ordinal number α , we define:

$$\begin{aligned} cl_X^0 A &= A \\ cl_X^1 A &= cl_X A \\ &\vdots \\ cl_X^\alpha A &= cl_X (cl_X^{\alpha-1} A) \text{ if } \alpha-1 \text{ exists} \\ cl_X^\alpha A &= \bigcup_{\beta < \alpha} cl_X^\beta A, \text{ if } \alpha \text{ is a limit ordinal.} \end{aligned}$$

The smallest ordinal α such that $cl_X^\alpha A = cl_X^{\alpha+1} A$ for all $A \subseteq X$ is called the length of the decomposition series of X and denoted by $\ell_D(X)$. The relationship between $\ell_D(X)$ and $\ell_D(X^*)$ can be obtained with the help of several lemmas.

For the remainder of this section, we shall assume that X is an arbitrary space; (X^*, i^*) will always denote the star compactification of X . Let ω be the smallest infinite ordinal number.

LEMMA 2.4. If $A \subseteq X$, then $cl_{X^*}^n A = cl_X^n A \cup (cl_X^{n-1} A)'$.

PROOF. It suffices to prove this result for $n=2$. Note that $\text{cl}_X^2 A \cup (\text{cl}_X A)' \subseteq \text{cl}_{X^*}^2 A$ is obvious. If $x \in (\text{cl}_{X^*}^2 A) \cap X$, then there is $G \in \mathcal{U}(X^*)$ such that $G \geq \mathcal{L}^*$, and $x \in G$ implies $G|_X \geq \mathcal{L}$. Thus $x \in \text{cl}_X^2 A$. If $\mathcal{F} \in \text{cl}_{X^*}^2 A \cap X'$, then it is easy to see that $\text{cl}_X A \in \mathcal{F}$, and so $\mathcal{F} \in (\text{cl}_X A)'$. ■

We next describe $\text{cl}_{X^*}^n B$ for $B \subseteq X'$. For this purpose, it is necessary to introduce some additional terminology. If $B \subseteq X'$, let $\mathcal{H}_B = \{\mathcal{F} : \mathcal{F} \in B\}$; note that $\mathcal{H}_B \in F(X)$. Define $B^\sim = \{\mathcal{L} \in X' : \mathcal{L} \geq \mathcal{H}_B\}$, and $B^\vee = \{x \in X : \exists \mathcal{F} \in \mathcal{U}(X) \text{ such that } \mathcal{F} \rightarrow x \text{ in } X \text{ and } \mathcal{F} \geq \mathcal{H}_B\}$. Note that $(B^\sim)^\sim = B^\sim$ and $(B^\sim)^\vee = B^\vee$.

LEMMA 2.5. If $B \subseteq X'$, then $\text{cl}_{X^*}^n B = B^\sim \cup (\text{cl}_X^{n-1} B^\vee) \cup (\text{cl}_X^{n-2} B^\vee)'$.

PROOF. For $n=1$, the statement becomes $\text{cl}_{X^*} B = B^\sim \cup B^\vee$. If $\mathcal{L} \in (\text{cl}_{X^*} B) \cap X'$, then there is $G \in \mathcal{U}(X^*)$ such that $G \rightarrow \mathcal{L}$ in X^* and $B \in G$. Thus $G \geq \mathcal{L}'$, and so $B \cap G' \neq \emptyset$ for all $G \in \mathcal{L}$. This implies $\mathcal{L} \geq \mathcal{H}_B$, and so $\mathcal{L} \in B^\sim$. If $x \in (\text{cl}_{X^*} B) \cap X$, then there is an u.f. G containing B and a filter $\mathcal{F} \rightarrow x$ such that $G \geq \mathcal{F}'$. By Proposition 2.2, $G^\wedge \geq \mathcal{F}$, and so $G^\wedge \rightarrow x$ in X . Also, $G^\wedge \in \mathcal{U}(X)$ and $G \geq (G^\wedge)'$. Letting $\mathcal{L} = G^\wedge$, we have $B \cap G' \neq \emptyset$ for all $G \in \mathcal{L}$. Thus $\mathcal{L} \geq \mathcal{H}_B$, and $x \in B^\vee$.

Conversely, if $x \in B^\vee$, then there is $\mathcal{F} \in \mathcal{U}(X)$ such that $\mathcal{F} \rightarrow x$ and $\mathcal{F} \geq \mathcal{H}_B$. For each $F \in \mathcal{F}$, choose $\mathcal{L}_F \in B$ such that $F \in \mathcal{L}_F$. Let G be the filter of sections of the net $(\mathcal{L}_F)_{F \in \mathcal{F}}$. Then $G \geq \mathcal{F}^*$ implies $G \rightarrow x$ in X^* , and so $x \in \text{cl}_{X^*} B$. A similar argument shows that $B^\sim \subseteq \text{cl}_{X^*} B$. This establishes the result for $n=1$.

If $n=2$, then $\text{cl}_{X^*}^2 B = \text{cl}_{X^*} (B^\sim \cup B^\vee) = (B^\sim)^\sim \cup (B^\sim)^\vee \cup (\text{cl}_X B^\vee)'$ follows with the help of Lemma 2.4. By the remarks preceding Lemma 2.5, $(B^\sim)^\sim = B^\sim$ and $(B^\sim)^\vee = B^\vee = B^\vee \subseteq \text{cl}_X B^\vee$. This establishes the result for $n=2$. The generalization to arbitrary n is now a trivial induction argument.

COROLLARY 2.6. If $A \subseteq X$, then $\text{cl}_{X^*} A = \text{cl}_{X^*} A^*$.

PROOF. $cl_{X*} A^* = cl_{X*} A \cup cl_{X*} A'$. By Lemma 2.5, $cl_{X*} A' = (A')^\sim \cup (A')^\vee$.

It is easy to check that $(A')^\sim = A' \subseteq cl_{X*} A$, and $(A')^\vee \subseteq cl_X A \subseteq cl_{X*} A$. ■

LEMMA 2.7 (a) If $A \subseteq X$ and α is a non-limit ordinal, then

$$cl_{X*}^\alpha A = cl_X^\alpha A \cup (cl_X^{\alpha-1} A)'$$

(b) If α is a limit ordinal, then $cl_{X*}^\alpha A = cl_X^\alpha A \cup (U\{(cl_X^\beta A)'\} : \beta < \alpha)$.

PROOF. Using transfinite induction along with Lemma 2.4, the results follow easily from both limit and non-limit ordinals, except for the case $\alpha = \gamma + 1$, where γ is a limit ordinal. In this case we have

$$cl_{X*}^{\gamma+1} A = cl_{X*}(cl_X^\gamma A \cup B) = cl_X^{\gamma+1} A \cup (cl_X^\gamma A)' \cup B^\sim \cup B^\vee,$$

where $B = U\{(cl_X^\beta A)'\} : \beta < \alpha\}$. It is not difficult to show that $B^\vee \subseteq cl_X^{\gamma+1} A$ and $B^\sim \subseteq (cl_X^\gamma A)'$; we omit the details. ■

The symbol λX represents the topological modification of X , i.e., the finest topological space on X coarser than X .

THEOREM 2.8. λX is a subspace of λX^* .

PROOF. Let $X_1 = \lambda X^*|_X$. Then $\lambda X \geq X_1$ is clear. Let $A \subseteq X$ be λX -closed. Then $cl_{X*} A = cl_X A \cup A' = A \cup A' = A^* = cl_X A^*$, by Corollary 2.6. Thus A^* is closed in λX^* and $A = A^* \cap X$, which implies A is X_1 -closed. ■

THEOREM 2.9. (a) If $1 \leq \ell_D(X) < \omega$, then $\ell_D(X) \leq \ell_D(X^*) \leq \ell_D(X) + 1$.

(b) If $\ell_D(X) \geq \omega$, then $\ell_D(X) = \ell_D(X^*)$.

PROOF. Let $A \subseteq X^*$ and $A = B \cup C$, where $B = A \cap X$, $C = A \cap X'$. Then, by Lemmas 2.4 and 2.5, $cl_{X*}^n A = cl_X^n B \cup (cl_X^{n-1} B)' \cup (cl_X^{n-1} C^\vee) \cup (cl_X^{n-2} C^\vee) \cup C^\sim$. If cl_X^k is idempotent, it follows that cl_{X*}^{k+1} must be idempotent. Thus $\ell_D(X^*) \leq \ell_D(X) + 1$. It follows easily from Lemma 2.4 that cl_X^k is idempotent if cl_{X*}^k is idempotent. Thus (a) is established.

Statement (b) follows easily from Lemma 2.6. ■

COROLLARY 2.10. If X is a topological space, then $\ell_D(X^*) \leq 2$.

We define a G-space to be a regular space with the property that $\text{cl}_X \mathfrak{F} = \mathfrak{F}$ for all $\mathfrak{F} \in X'$; this concept (but not the terminology) was introduced by Gazik [4]. Discrete spaces and compact regular spaces are the most obvious examples of G-spaces. Another class of G-spaces are the ultraspace, which are the topological spaces having exactly one convergent ultrafilter. The next theorem is proved in [4] in the case where X is T_2 ; removal of the T_2 assumption causes no problems in the proof.

THEOREM 2.11. X^* is regular iff X is a G-space.

A regular space is symmetric if $\mathfrak{F} \rightarrow x$ whenever $\mathfrak{F} \rightarrow y$ and $\dot{y} \rightarrow x$. Examples of symmetric spaces include T_3 spaces, regular topological spaces, and regular convergence groups. It is shown in [14] that a compact symmetric space has the same ultrafilter convergence as a compact regular topological space.

PROPOSITION 2.12. If X is a symmetric G-space, then X^* is symmetric.

PROOF. Let $\dot{G} \rightarrow \alpha$ and $\dot{\alpha} \rightarrow \beta$ in X^* . By the construction of X^* , we can conclude that both α and β are in X . Since $\dot{G} \rightarrow \alpha$, there is $\mathfrak{F} \rightarrow \alpha$ in X such that $\dot{G} \geq \mathfrak{F}^*$. Since X is symmetric, $\mathfrak{F} \rightarrow \beta$ in X , and so $\dot{G} \rightarrow \beta$ in X^* . ■

THEOREM 2.13. (a) If X is a G-space, then $\ell_D(X^*) \leq 2$ and $\ell_D(X) \leq 2$.

(b) If X is a symmetric G-space, then $\ell_D(X^*) \leq 1$.

(c) If X is a T_2 topological space, then $\ell_D(X^*) \leq 1$ iff X is a G-space.

PROOF. (a) If X is a G-space, then X^* is a compact regular space by Theorem 2.11, and it follows by Theorem 2.4(a), [14], that $\ell_D(X^*) \leq 2$. The second inequality follows from the first and Theorem 2.9.

Statement (b) follows immediately from Proposition 2.12 and Theorem 2.4(b), [14].

(c) If X is a topological G-space, then X is symmetric, and so $\ell_D(X^*) \leq 1$ by (b). If $\ell_D(X^*) \leq 1$, then X^* is a compact T_2 topological space, and hence X^* is regular. By Theorem 2.11, X is a G-space. ■

It should be noted that $\ell_D(Y) = 0$ iff Y is discrete, and consequently $\ell_D(X^*) = 0$ iff X is a finite discrete space. If X is not a finite discrete space, we can replace " $\ell_D(X^*) \leq 1$ " by " $\ell_D(X^*) = 1$ " in (b) and (c) of Theorem 2.13.

3. R-SERIES.

In this section, we summarize some results on the R-series of a space (originally studied in [13] and [14]), obtain a few results on the R-series of X^* , and lay the groundwork for many of the results of Section 4.

Starting with a space X , an ordinal family of spaces $\{r_\alpha X\}$ is defined on the same underlying set as follows: $r_0(X) = X$

$\mathfrak{F} \rightarrow x$ in $r_1 X$ iff there exist $n \in \mathbb{N}$ and $\mathfrak{G} \rightarrow x$ in X such that $\mathfrak{F} \geq \text{cl}_X^n \mathfrak{G}$.

$\mathfrak{F} \rightarrow x$ in $r_\alpha X$ iff there exist $n \in \mathbb{N}$, $\mathfrak{G} \rightarrow x$ in X and $\beta < \alpha$ such that $\mathfrak{F} \geq \text{cl}_{r_\beta X}^n \mathfrak{G}$.

The family $\{r_\alpha X\}$ is called the R-series of X . If γ is the least ordinal such that $r_\gamma X = r_{\gamma+1} X$, then $r_\gamma X$ is denoted X_r , and γ is called the length of the R-series of X and denoted by $\ell_R(X)$. Note that X is regular iff $\ell_R(X) = 0$. It is shown in [13] that X_r is the finest regular space coarser than X , and X_r is called the regular modification of X .

Of course, X_r will not in general be T_2 , even when X is T_2 . A T_3 space associated with X (we use T_3 to mean regular plus T_1) is constructed as follows. First, define an equivalence relation among the elements of X by: $x \sim y$ iff $\dot{x} \rightarrow y$ in X_r . Let X_s be the set of equivalence classes with the quotient convergence structure derived from X_r .

If $f: X \rightarrow Y$, let $\bar{f}: X_s \rightarrow Y_s$ be the (unique) function which makes the following diagram commute:

$$\begin{array}{ccccc}
 X & \longrightarrow & X_r & \xrightarrow{\varphi_X} & X_s \\
 \downarrow f & & \downarrow f & & \downarrow \bar{f} \\
 Y & \longrightarrow & Y_r & \xrightarrow{\varphi_Y} & Y_s
 \end{array}$$

where the maps from X to X_r and Y to Y_r are the respective identity maps, and ϕ_X , ϕ_Y are the respective quotient maps.

Let σX denote the symmetric modification of X , i.e., the finest symmetric space coarser than X . The next proposition follows immediately from results of [13] and [14].

PROPOSITION 3.1. If $f: X \rightarrow Y$ is continuous, then each map in the following commutative diagram (in which all non-labeled vertical maps are f and all non-labeled horizontal maps are identities) is continuous.

$$\begin{array}{ccccccccccc}
 X & \rightarrow & R_1 X & \rightarrow & \dots & \rightarrow & r_\alpha X & \rightarrow & \dots & \rightarrow & X_r & \rightarrow & \sigma X & \xrightarrow{\phi_X} & X_s \\
 \downarrow & & \downarrow & & & & \downarrow & & & & \downarrow & & \downarrow & \phi_Y & \downarrow \bar{f} \\
 Y & \rightarrow & r_1 Y & \rightarrow & \dots & \rightarrow & r_\alpha Y & \rightarrow & \dots & \rightarrow & Y_r & \rightarrow & \sigma Y & \rightarrow & Y_s
 \end{array}$$

PROPOSITION 3.2. For any space X , $r_1 X$ is a subspace of $r_1 X^*$.

PROOF. Let X_1 be the restriction of $r_1 X^*$ to X . Since $i^*: X \rightarrow X^*$ is continuous, it follows from Proposition 3.1 that $i^*: r_1 X \rightarrow r_1 X^*$ is continuous, and thus $r_1 X \geq X_1$. On the other hand, let $\mathcal{F} \rightarrow x$ in X_1 . Then there is $\mathcal{G} \rightarrow x$ in X^* and $n \in \mathbb{N}$ such that $\mathcal{F} \geq \text{cl}_{X^*}^n \mathcal{G}$. Since $\mathcal{G} \rightarrow x$ in X^* , there is $\mathcal{H} \rightarrow x$ in X such that $\mathcal{G} \geq \mathcal{H}^*$, and so we have $\mathcal{F} \geq \text{cl}_{X^*}^n \mathcal{H}^* = \text{cl}_{X^*}^n \mathcal{H} = \text{cl}_X^n \mathcal{H} \cap (\text{cl}_X^{n-1} \mathcal{H}')$, by Lemma 2.4 and Corollary 2.6. Since $\mathcal{F} \in F(X)$, $\mathcal{F} \geq \text{cl}_X^n \mathcal{H}$, and so $\mathcal{F} \rightarrow x$ in $r_1 X$. ■

PROPOSITION 3.3. If X is a locally compact T_3 space, then $r_2 X$ is a subspace of $r_2 X^*$.

PROOF. It is sufficient to show that for any $A \subseteq X$, $\text{cl}_{r_1 X}^n A = (\text{cl}_{r_1 X^*}^n A) \cap X$ for all $n \in \mathbb{N}$, and this will be proved by induction. For $n=1$, the equality follows by Proposition 3.2. If the equality holds for n and $x \in (\text{cl}_{r_1 X^*}^{n+1} A) \cap X$, then there is $\mathcal{F} \rightarrow x$ and $k \in \mathbb{N}$ such that $(\text{cl}_{X^*}^k \mathcal{F}) \cap (\text{cl}_{r_1 X^*}^n A) \neq \emptyset$ for all $\mathcal{F} \in \mathcal{F}$. Since X is locally compact and T_3 , we may assume without loss of

generality that $\mathfrak{F}$ has a filter base of compact sets, and so $\text{cl}_{X^*}^k \mathfrak{F} = \text{cl}_X^k \mathfrak{F}$.

From this observation, along with the induction hypothesis, we may conclude that $x \in \text{cl}_{r_1 X}^{n+1} A$. ■

It is not true in general that $r_\alpha X$ is a subspace of $r_\alpha X^*$. To establish this fact, we need to make use of some theorems from [9].

A space X is defined to be completely regular if it is a subspace of a symmetric compact space. It is shown in [9] that X is completely regular iff it is a symmetric space with the same ultrafilter convergence as a completely regular topological space. Let ωX denote the finest completely regular space coarser than X . A space X is defined to be ω -regular if $\mathfrak{F} \rightarrow x$ implies $\text{cl}_{\omega X} \mathfrak{F} \rightarrow x$. It is proved in [9] that X is ω -regular iff X is a subspace of a compact regular space. The ω -regular spaces include the completely regular spaces and also the c -embedded spaces of Binz [1].

PROPOSITION 3.4. (a) If X is a regular space which is not ω -regular, then X_r is not a subspace of X_r^* , and $\ell_R(X^*) \geq 2$.

(b) If X is a locally compact T_3 space which is not ω -regular, then $\ell_R(X^*) \geq 3$.

PROOF. The first part of (a) follows from the aforementioned characterization of ω -regular spaces as subspaces of compact regular spaces. The two statements concerning $\ell_R(X^*)$ follow by Proposition 3.2 and 3.3, respectively. ■

For any space X , let $C(X)$ be the set of all continuous real-valued functions on X . A T_3 topological space X for which $C(X)$ consists only of constant functions is an example of a regular space which is not ω -regular; for this space, $\ell_R(X) = 0$ and $\ell_R(X^*) \geq 2$. An example of a regular space X for which $\ell_R(X^*) \geq 3$ is obtained with the help of the following lemma.

LEMMA 3.5. A locally compact T_3 space X is ω -regular iff $C(X)$ separates points in X .

PROOF. If $X = \omega X$, then ωX is T_2 and so $C(X)$ separates points in X . Conversely, if $\mathcal{F} \rightarrow x$ in X , then $\mathcal{F}$ contains a compact set A . Since $C(X)$ separates points in X , ωX is T_2 and so the subspaces $X|_A$ and $\omega X|_A$ have the same ultrafilter convergence. It follows that $\text{cl}_X \mathcal{F} = \text{cl}_{\omega X} \mathcal{F} \rightarrow x$ in X , and therefore X is ω -regular. ■

EXAMPLE 3.6. Let X be the set $[0, 1]$. If s is a sequence on the set X , let $\mathcal{F}_s$ denote the filter generated by s in the usual way. Let $\mathcal{J}$ be the filter generated by the sequence $(\frac{1}{n})$. Define a convergence on X as follows:
 (1) If $x \notin \{0, 1\}$, then $\mathcal{F} \rightarrow x$ iff there is a sequence s converging to x in the usual topology such that $\mathcal{F} \geq \mathcal{F}_s$; (2) $\mathcal{F} \rightarrow 0$ iff there is a sequence s converging to 0 in the usual topology, but not a subsequence of $(\frac{1}{n})$, such that $\mathcal{F} \geq \mathcal{F}_s$; (3) $\mathcal{F} \rightarrow 1$ iff $\mathcal{F} \geq \mathcal{F}_s$, where s is a sequence converging to 1 in the usual topology, or else $\mathcal{F} \geq \mathcal{J} \cap 1$. One may easily verify that the space X is locally compact and T_3 , but $C(X)$ will not separate the points 0 and 1. Thus, by Lemma 3.5, X is not ω -regular, and it follows from Proposition 3.4 that $\ell_R(X^*) \geq 3$, whereas $\ell_R(X) = 0$; this result contrasts with the conclusion of Theorem 2.9. One can also show (we omit the details) that $\dot{0} \rightarrow 1$ in $r_3 X^*$. Since $r_3 X = X$, it follows that $r_3 X$ is not a subspace of $r_3 X^*$. This shows that the conclusion of Proposition 3.3 cannot be improved without imposing additional conditions. ■

Gazik showed in [4] that a T_3 pretopological G -space is a completely regular topological space. Another result along these lines is

PROPOSITION 3.7. (a) A symmetric G -space is completely regular.

(b) Every G -space is ω -regular.

PROOF. These statements follow immediately from Theorem 2.11, Proposition 2.12, and the characterization of ω -regular spaces obtained in [9]. ■

THEOREM 3.8. Let X be a space.

- (a) X is regular iff X is a subspace of $r_1 X^*$.
- (b) X is ω -regular iff X is a subspace of X_r^* .
- (c) X is completely regular iff X is a subspace of σX^* .

Proof. Statement (a) follows immediately from Proposition 3.2. Statements (b) and (c) are proved in [9]. ■

For topological spaces X and Y , function $f : X \rightarrow Y$ is defined to be θ -continuous if, for every $x \in X$ and every neighborhood V of $f(x)$, there is a neighborhood $\mathcal{U}$ of $f(x)$ such that $f(\text{cl}_X \mathcal{U}) \subseteq \text{cl}_Y V$.

PROPOSITION 3.9. Let $f : X \rightarrow Y$, where X and Y are topological spaces. Then $f : X \rightarrow Y$ is θ -continuous iff $f : r_1 X \rightarrow r_1 Y$ is continuous.

PROOF. Let $f : X \rightarrow Y$ be θ -continuous, and let $\mathcal{F} \rightarrow x$ in $r_1 X$. Then there is $\mathcal{G} \rightarrow x$ in X such that $\mathcal{F} \geq \text{cl}_X \mathcal{G} \geq \text{cl}_X \mathcal{U}(x)$, where $\mathcal{U}(x)$ is the neighborhood filter at x . By θ -continuity, $f(\text{cl}_X \mathcal{U}(x)) \geq \text{cl}_Y \mathcal{U}(f(x))$, and so $f(\mathcal{F}) \geq \text{cl}_Y \mathcal{U}(f(x))$. The latter filter $r_1 Y$ -converges to $f(x)$, and so $f : r_1 X \rightarrow r_1 Y$ is continuous. For the converse argument, one easily see that $f : r_1 X \rightarrow r_1 Y$ implies $f(\text{cl}_X \mathcal{U}(x)) \geq \text{cl}_Y \mathcal{U}(f(x))$ for all $x \in X$, which is equivalent to θ -continuity of $f : X \rightarrow Y$. ■

The characterization of θ -continuity given in Proposition 3.9 is not suitable for a purely topological investigation, since $r_1 X$ may fail to be topological even when X is topological. Perhaps this suggests that convergence spaces are the natural realm for the study of θ -continuity. But in any event, we shall define a function $f : X \rightarrow Y$ between arbitrary convergence spaces to be θ -continuous if $f : r_1 X \rightarrow r_1 Y$ is continuous.

More generally, if P is any function property, then $f : X \rightarrow Y$ is defined to have property θ -P if $f : r_1 X \rightarrow r_1 Y$ has property P . Thus, one can speak of θ -open maps, θ -quotient maps, etc. Some of these " θ -properties" will be studied

briefly in Section 6.

4. NATURAL EXTENSIONS.

Let X and Y be spaces and consider a function $f : X \rightarrow Y$. A function $f_* : X^* \rightarrow Y^*$ is called a natural extension of f if the following conditions are satisfied:

$$(1) \quad f_* \mid_X = f.$$

(2) If $\mathfrak{F} \in X'$ and $f(\mathfrak{F})$ is convergent in Y , then $f_*(\mathfrak{F})$ is an element of Y to which $f(\mathfrak{F})$ converges.

$$(3) \quad \text{If } \mathfrak{F} \in X' \text{ and } f(\mathfrak{F}) \in Y', \text{ then } f_*(\mathfrak{F}) = f(\mathfrak{F}) \in Y'.$$

If $\mathfrak{F} \in X'$ implies $f(\mathfrak{F}) \in Y'$, then f is said to be weakly proper. If $f : X \rightarrow Y$ is a weakly proper function, or if Y is T_2 , then the natural extension f_* is unique; in general, f may have many natural extensions. In the proposition and theorem that follow, when $f : X \rightarrow Y$, f_* will be assumed to be an arbitrary natural extension of f .

The proof of the next lemma is straightforward and will be omitted.

LEMMA 4.1. If $f : X \rightarrow Y$ and $A \subseteq X$, then :

- (a) If f is continuous, then $(f(A))^* \subseteq f_*(A^*) \subseteq f_*(cl_{X^*} A) \subseteq cl_{Y^*} f(A)$;
- (b) If f is continuous and Y is T_2 , then $f_*(cl_{X^*} A) = cl_{Y^*} f(A)$;
- (c) If f is weakly proper, then $f_*(A^*) \subseteq (f(A))^*$.

THEOREM 4.2. If $f : X \rightarrow Y$ is continuous, then $f_* : X^* \rightarrow Y^*$ is θ -continuous.

PROOF. It is sufficient to show that, for each $\mathfrak{F} \in F(X)$, $f_*(cl_{X^*}^n \mathfrak{F}^*) \geq cl_{Y^*}^n f(\mathfrak{F})$. If $F \in \mathfrak{F}$, then $cl_{X^*}^n F = cl_{X^*}^n F^* = cl_X^n F \cup (cl_X^{n-1} F)'$ by Lemma 2.4 and Corollary 2.6. By continuity of f , $f_*(cl_X^n F) = f(cl_X^n F) \subseteq cl_Y^n f(F)$, and $f_*(cl_X^{n-1} F)^* \subseteq cl_{Y^*} f(cl_X^{n-1} F) \subseteq cl_{Y^*} (cl_Y^{n-1} f(F)) \subseteq cl_{Y^*}^n f(F)$ follows with the help of Lemma 4.1. Thus $f_*(cl_{X^*}^n F^*) \subseteq cl_{Y^*}^n f(F)$, and the theorem is proved. ■

COROLLARY 4.3. If $f : X \rightarrow Y$ is continuous, then each map in the following commutative diagram (in which all non-labeled vertical maps are f_* and all non-labeled horizontal maps are identities) is continuous.

$$\begin{array}{ccccccccc}
 r_1 X^* & \rightarrow & \dots & \rightarrow & r_\alpha X^* & \rightarrow & \dots & \rightarrow & X^* r & \rightarrow & \sigma X^* & \xrightarrow{\phi_{X^*}} & X^* s \\
 \downarrow & & & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow (f_*)^- \\
 r_1 Y^* & \rightarrow & \dots & \rightarrow & r_\alpha Y^* & \rightarrow & \dots & \rightarrow & Y^* r & \rightarrow & \sigma Y^* & \xrightarrow{\phi_{Y^*}} & Y^* s
 \end{array}$$

The next result closely resembles, but is more general than, the extension property of the star compactification obtained in [11].

COROLLARY 4.4. If $f : X \rightarrow Y$ is continuous and Y is compact and regular, then $f_* : X^* \rightarrow Y$ is continuous. If Y is also T_2 , then the extension f_* is unique.

PROOF. Under the given assumptions, $Y = Y^* = r_1 Y^*$; since $X^* \geq r_1 X^*$, the first statement is established. The second follows from an earlier remark. ■

In the next section we shall see that continuity of $f : X \rightarrow Y$ does not guarantee the continuity of $f_* : X^* \rightarrow Y^*$. If X is a regular space, let $X^\sim = r_1 X^*$; then by Proposition 3.2 $\kappa^\sim = (X^\sim, i^*)$ is a compactification of X . Our study of the compactification $\kappa^\sim$ will be limited to the following proposition.

PROPOSITION 4.5. Let X and Y be regular spaces.

- (a) If $f : X \rightarrow Y$ is continuous, then $f_* : X^\sim \rightarrow Y^\sim$ is continuous.
- (b) $X^\sim$ is T_2 iff X is a T_2 G-space.

PROOF. Statement (a) follows immediately from Theorem 4.2.

(b) If X is a T_2 G-space, then X^* is regular by Proposition 2.10, and so $X^\sim = X^*$ is T_2 . If X is not a G-space, then there is $\mathfrak{F} \in X'$ such that $\mathfrak{F} > \text{cl}_X \mathfrak{F}$, where $\mathfrak{F} \in \mathcal{U}(X)$ and $\mathfrak{F} \neq \mathfrak{F}$. If $\mathfrak{F} \xrightarrow{X} x$, then $x \rightarrow \mathfrak{F}$ in $X^\sim$. If $\mathfrak{F} \in X'$, then the filter $\mathfrak{F}_1$ on X^* generated by $\mathfrak{F}$ converges in X^* to both $\mathfrak{F}$ and $\mathfrak{F}$. ■

It is shown in [12] that every completely regular T_2 space has a Stone-Ćech compactification. This compactification is regular and T_2 , has the universal property relative to the class of completely regular spaces, and agrees with the

COROLLARY 4.7. If X is a completely regular T_2 space, then (X_s^*, i_s^-) is the Stone-Čech compactification of X .

If X is a Tychonoff topological space, then Corollary 4.7 gives a new method for constructing βX . Indeed, βX is in this case the pretopological modification of X_s^* .

5. CONTINUITY OF NATURAL EXTENSIONS.

We next consider conditions under which a natural extension f_* of a continuous function f is continuous. For this purpose, we use some additional notation and terminology.

Let $f: X \rightarrow Y$ be a continuous function, and let $f_*: X^* \rightarrow Y^*$ be a natural extension of f . For $A \subseteq X$, define $A'_f = \{ \mathfrak{F} \in A' : f(\mathfrak{F}) \text{ converges in } Y \}$. Let $\Gamma_{f_*}(A) = f(A) \cup f_*(A'_f)$; note that $\Gamma_{f_*}(A) = f_*(A^*) \cap Y$. If $\mathfrak{F} \in F(X)$ define $\Gamma_{f_*}(\mathfrak{F}) \in F(Y)$ to be the filter generated by $\{ \Gamma_{f_*}(F) : F \in \mathfrak{F} \}$; $\mathfrak{F} \in F(X)$ is said to be f_* -closed if $\Gamma_{f_*}(\mathfrak{F}) = f(\mathfrak{F})$.

PROPOSITION 5.1. Let f be a continuous map.

- (1) f_* is continuous at $x \in f^{-1}(Y)$ iff $\mathfrak{F} \rightarrow x$ in X implies that $\Gamma_{f_*} \mathfrak{F} \rightarrow f(x)$ in Y .
- (2) f_* is continuous at $\mathfrak{F} \in f_*^{-1}(Y) \cap X$ iff $\Gamma_{f_*}(\mathfrak{F}) \rightarrow f_* \mathfrak{F}$ in Y .
- (3) f_* is continuous at $\mathfrak{F} \in f_*^{-1}(Y')$ iff $\mathfrak{F}$ is f_* -closed.

PROOF. If $\mathfrak{F} \in F(X)$ then one can easily show that $\Gamma_{f_*} \mathfrak{F} \geq (\Gamma_{f_*}(\mathfrak{F}))^*$; (1) and (2) follow from these inequalities.

(3) If f_* is continuous at $\mathfrak{F} \in f_*^{-1}(Y')$, then $\Gamma_{f_*} \mathfrak{F} \geq f_*(\mathfrak{F}^*) \geq (f(\mathfrak{F}))^*$, and hence $\Gamma_{f_*} \mathfrak{F} = f(\mathfrak{F})$. Conversely, $f_*(\mathfrak{F}^*) \geq (\Gamma_{f_*} \mathfrak{F})^* = (f \mathfrak{F})^* \rightarrow f_* \mathfrak{F}$ in Y^* , and thus f_* is continuous at $\mathfrak{F} \in f_*^{-1}(Y')$. ■

COROLLARY 5.2. Let $f: X \rightarrow Y$ be continuous, and Y a regular space. Then

- (1) f_* is continuous at all points of $f_*^{-1}(Y)$.
- (2) f_* is continuous iff each $\mathfrak{F} \in f_*^{-1}(Y')$ is f_* -closed.

topological Stone-Čech compactification relative to ultrafilter convergence when X is topological. We shall now give an alternate construction of this compactification using κ^* .

For any space X , X_s^* is a compact, regular, T_2 space. However it is not generally true that X_s is a subspace of X_s^* . Recall the notation σX for the symmetric modification of X .

THEOREM 4.6. If X is a space such that σX is a subspace of σX^* , then σX is completely regular, X_s is completely regular and T_2 , and $(X_s^*, i_\star^-)$ is the Stone-Čech compactification of X_s .

PROOF. By assumption, σX is a subspace of a compact symmetric space, and hence completely regular. X_s is T_2 by construction. In the diagram that follows

$$\begin{array}{ccc} \sigma X & \xrightarrow{i_\star^-} & \sigma X^* \\ \phi_X \downarrow & & \downarrow \phi_{X^*} \\ X_s & \xrightarrow{i_\star^-} & X_s^* \end{array}$$

the maps ϕ_X and ϕ_{X^*} are strongly open (see Proposition 2.2, [14]). This means that if $\mathcal{L} \rightarrow \alpha$ in X_s and $x \in \phi_X^{-1}(\alpha)$, then there is a filter $\mathcal{F}$ on σX such that $\mathcal{F} \rightarrow x$ in σX and $\phi_X(\mathcal{F}) = \mathcal{L}$. Using this property and the fact that σX and σX^* are symmetric, one can easily show that X_s is densely embedded in X_s^* .

If Y is a regular, compact, T_2 space and $f : X_s \rightarrow Y$ is continuous, then define $F : X \rightarrow Y$ by $F(x) = f([x])$, where $[x]$ is the equivalence class in X_s defined by x . It is easy to check that $F : X \rightarrow Y$ is continuous, and so by Corollary 4.3, $(F_\star)^- : X_s^* \rightarrow Y_s^* = Y$ is continuous. $(F_\star)^-$ is clearly an extension of f , and this extension is unique because Y is Hausdorff. Thus by the uniqueness of the Stone-Čech compactification established in [12], it follows that this compactification is equivalent to $(X_s^*, i_\star^-)$. ■

(3) If X is essentially bounded, then f_* is continuous.

PROOF. Statements (1) and (2) follow immediately from Proposition 5.1 and the fact that Y is regular. The assumption that X is essentially bounded (see Section 2 for this definition) guarantees that each $\mathfrak{F} \in f_*^{-1}(Y')$ is f_* -closed. ■

PROPOSITION 5.3. If $f : X \rightarrow Y$ is continuous and weakly proper, then f_* is continuous.

PROOF. If f is weakly proper and $A \subseteq X$, then $A'_f = \emptyset$. Thus each filter $\mathfrak{F} \in F(X)$ is f_* -closed. The conditions (1) and (3) of Proposition 5.1 for continuity of f_* are thus satisfied, while condition (2) is satisfied vacuously. ■

COROLLARY 5.4. If X is a closed subspace of Y and $f : X \rightarrow Y$ is the identity embedding, then $f_* : X^* \rightarrow Y^*$ is also an embedding.

PROOF. Since X is closed in Y , f is weakly proper; thus f_* is continuous by Proposition 5.3. f_* is clearly one-to-one, and by Lemma 4.1, $f_*(\mathfrak{F}^*) = f(\mathfrak{F})^*$ for all $\mathfrak{F} \in F(X)$. From this equality, it follows easily that f_* is an embedding. ■

PROPOSITION 5.5. The following statements about a regular space Y are equivalent.

- (a) Y is a G -space.
- (b) Every natural extension of every continuous function into Y is continuous.
- (c) If Z and Y have the same set, Z is discrete, and $f : Z \rightarrow Y$ is the identity, then f_* is continuous.

PROOF. (a) $\Rightarrow$ (b). If $f : X \rightarrow Y$ is continuous and Y is a G -space, then Y^* is regular by Theorem 2.11 and so $f_* : X^* \rightarrow Y^*$ is continuous by Corollary 4.4.

(b) $\Rightarrow$ (c). Obvious.

(c) $\Rightarrow$ (a). If Y is not a G -space, then there is $\mathfrak{F} \in Y'$ such that $\text{cl}_Y \mathfrak{F} \neq \mathfrak{F}$. Since Y is T_1 and Z is discrete, it follows that $\Gamma_{f_*} \mathfrak{F} \neq \mathfrak{F}$ for some natural extension f_* . Thus $\mathfrak{F}$ is not f_* -closed, and by Proposition 5.1 f_* is not

continuous. ■

The final result in this section is analogous to Proposition 5.1, but involves θ -continuity rather than continuity. Since this result is of marginal interest, we shall omit the proof.

PROPOSITION 5.6. Let $f : X \rightarrow Y$ be a θ -continuous map.

- (a) f_* is θ -continuous at each point $x \in X$.
- (b) f_* is θ -continuous at $\mathcal{F} \in f_*^{-1}(Y) \cap X'$ iff $f(\text{cl}_X^n \mathcal{F}) \rightarrow f_* \mathcal{F}$ in $r_1 Y$ for each $n \geq 1$.
- (c) f_* is θ -continuous at $\mathcal{F} \in f_*^{-1}(Y')$ iff, for each $n \geq 1$, there is $m \geq 1$ such that $f(\text{cl}_X^n \mathcal{F}) \geq \text{cl}_Y^m f(\mathcal{F})$.

6. QUOTIENT EXTENSIONS.

In this concluding section, we shall consider the circumstances under which f_* will possess certain quotient-type properties. We begin with definitions of the properties to be considered.

The term map will be used to mean a continuous, onto function. Note that if $f : X \rightarrow Y$ is onto, then any natural extension $f_* : X^* \rightarrow Y^*$ is also onto.

1. f is proper if f is a map and, whenever $\mathcal{L} \rightarrow y$ in Y , and $\mathcal{F}$ is an u.f. on X such that $f(\mathcal{F}) = \mathcal{L}$, then there is $x \in f^{-1}(y)$ such that $\mathcal{F} \rightarrow x$.
2. f is a convergence quotient map if f is a map and, whenever $\mathcal{L} \rightarrow y$ in Y , there is $x \in f^{-1}(y)$ and $\mathcal{F} \rightarrow x$ in X such that $f(\mathcal{F}) = \mathcal{L}$.
3. f is perfect if f is a proper convergence quotient map.
4. f is open if f is a map and whenever $\mathcal{L}$ is an u.f. on Y which converges to y , and $x \in f^{-1}(y)$, then there is an u.f. $\mathcal{F} \rightarrow x$ such that $f(\mathcal{F}) = \mathcal{L}$.
5. f is closure-preserving if $A \subseteq X$ implies $f(\text{cl}_X A) = \text{cl}_Y f(A)$.

Further information about these properties may be found in [6], [8], and [10]. Recall that if P represents any of the above properties, then $f : X \rightarrow Y$ is said to have property θ -P if $f : r_1 X \rightarrow r_1 Y$ has property P .

It is clear that a proper map is weakly proper, and that a weakly proper map onto a T_2 space is proper.

In all of the propositions that follow, f denotes a function from a space X into a space Y , and $f_* : X^* \rightarrow Y^*$ denotes a natural extension of f .

PROPOSITION 6.1. If f is a proper map, then f_* is also a proper map.

PROOF. Suppose that $\mathcal{G}$ is an u.f. on X^* such that $f_*(\mathcal{G}) \rightarrow \alpha$ in Y^* . Then there exists an u.f. $\mathcal{K}$ on Y such that $f_*(\mathcal{G}) \geq \mathcal{K}^*$, where $\mathcal{K}^* \rightarrow \alpha$ in Y^* . Suppose that $\mathcal{G} \rightarrow \beta$ in X^* ; then there exists a filter $\mathcal{L}$ on X such that $\mathcal{L} \rightarrow \beta$ in X^* and $\mathcal{G} \geq \mathcal{L}^*$. Hence $f_*(\mathcal{G}) \geq f_*(\mathcal{L}^*) = f(\mathcal{L})^*$ since f is a proper map, and thus $f(\mathcal{L})^*$ and $\mathcal{K}^*$ are not disjoint filters on Y^* . This implies that $f(\mathcal{L})$ and $\mathcal{K}$ are not disjoint filters on Y ; consequently, $f(\mathcal{L}) = \mathcal{K}$.

If $\alpha \notin Y$, then it follows that $f_*(\beta) = \alpha$. If $\alpha \in Y$, then since f is a proper map, $\mathcal{L} \rightarrow x$ in X for some $x \in f^{-1}(\alpha)$, and thus $\mathcal{G} \rightarrow x$ in X^* . It follows that f_* is a proper map. ■

PROPOSITION 6.2. If f is a convergence quotient map, then f_* is a convergence quotient map iff f_* is continuous.

PROOF. The relation $f(A)^* \subseteq f_*(A^*)$ is satisfied for each subset A of X , and hence $f_*(\mathcal{F}^*) \leq f(\mathcal{F})^*$ for each filter $\mathcal{F}$ on X . Suppose that $\mathcal{L}^* \rightarrow \alpha$ in Y^* and $\alpha \notin Y$; let $\mathcal{H}$ be any u.f. on X such that $f(\mathcal{H}) = \mathcal{L}$. Then $f_*(\mathcal{H}) = \alpha$ and $f_*(\mathcal{H}^*) \leq \mathcal{L}^*$. If $\alpha \in Y$, then there exists $\mathcal{F} \in F(X)$ and $x \in f^{-1}(\alpha)$ such that $\mathcal{F} \rightarrow x$ in X and $f(\mathcal{F}) = \mathcal{L}$, since f is a convergence quotient map. Since $f_*(\mathcal{F}^*) \leq \mathcal{L}^*$, it follows that f_* is a convergence quotient map precisely when f_* is a continuous map. ■

COROLLARY 6.3. f_* is a perfect map whenever f is a perfect map.

PROPOSITION 6.4. If f is an open, proper map, then f_* is open, proper, θ -open, and θ -proper.

PROOF. It follows by Proposition 6.1 that f_* is proper. If f_* is also open, then it follows from Theorem 4.2, [14], that f_* is also σ -open and σ -proper. Thus it remains only to show that f_* is open.

Let $G \in \mathcal{U}(Y^*)$ such that $G \rightarrow \alpha$ in Y^* , and let $\beta \in f_*^{-1}(\alpha)$. Then there is $\mathcal{J} \in \mathcal{U}(Y)$ such that $G \geq \mathcal{J}^*$ and $\mathcal{J}^* \rightarrow \alpha$ in Y^* . If $\alpha \notin Y$, then $\beta \notin X$, and hence there exists exactly one u.f. $\mathcal{F}$ on X such that $\mathcal{F} \rightarrow \beta$ in X^* . Thus $f(\mathcal{F}) = \mathcal{J}$, and since f is a proper map, $f_*(\mathcal{F}^*) = \mathcal{J}^* \leq G$. Let $\mathcal{B} \in \mathcal{U}(X^*)$ contain both $\mathcal{F}^*$ and $f_*^{-1}(G)$; then $f_*(\mathcal{B}) = G$, and $\mathcal{B} \rightarrow \beta$ in X^* .

If $\alpha \in Y$ then, since f is a proper map, $\beta \in X$, and since f is an open map, there is $\mathcal{F} \in \mathcal{U}(X)$ such that $\mathcal{F} \rightarrow \beta$ in X and $f(\mathcal{F}) = \mathcal{J}$. Then, as in the argument of the preceding paragraph, there exists an u.f. $\mathcal{B} \rightarrow \beta$ in X^* such that $f_*(\mathcal{B}) = G$ this establishes that f_* is an open map. ■

We omit the straightforward proof the next proposition.

PROPOSITION 6.5. If f is perfect θ -proper map, then f_* is a θ -perfect map.

Propositions 6.4 and 6.5 yield the following corollary.

COROLLARY 6.6. If f is an open perfect map, then f_* is a θ -perfect map.

PROPOSITION 6.7. If f is a convergent quotient map which is closure preserving, then f_* is a θ -convergence quotient map.

PROOF. Suppose that $G \rightarrow \alpha$ in $r_1 Y^*$. Then there is $\mathcal{J} \in \mathcal{F}(Y)$ such that $\mathcal{J}^* \rightarrow \alpha$ and $G \geq \text{cl}_Y^n \mathcal{J}$. If $\alpha \notin Y$, then it may be assumed that $\alpha = \mathcal{J} \in Y'$. Let $\mathcal{F} \in \mathcal{U}(X)$ such that $f(\mathcal{F}) = \mathcal{J}$; then $\beta = \mathcal{F} \in X'$ and $f_*(\beta) = \alpha$. By Lemma 2.4 and the assumption that f is closure-preserving, it follows that $f_*(\text{cl}_X^{n+1} \mathcal{F}^*) \leq f_*((\text{cl}_X^n \mathcal{F})^*) \leq \text{cl}_Y^n f(\text{cl}_X^n \mathcal{F}) = \text{cl}_Y^n \text{cl}_Y^n f(\mathcal{F}) \leq \text{cl}_Y^n \mathcal{J}^*$. If $\alpha \in Y$ then, since f is a convergence quotient map, there is $\mathcal{F} \rightarrow \beta \in f_*^{-1}(\alpha)$ such that $f(\mathcal{F}) = \mathcal{J}$. Again, $f_*(\text{cl}_X^{n+1} \mathcal{F}) \leq \text{cl}_Y^n \mathcal{J}^*$. In both cases $\text{cl}_X^{n+1} \mathcal{F}^* \rightarrow \beta$ in $r_1 X^*$. If $\mathcal{K} = f_*^{-1}(G) \vee \text{cl}_X^{n+1} \mathcal{F}$, then $\mathcal{K} \rightarrow \beta$ in $r_1 X^*$ and $f_*(\mathcal{K}) = G$. Thus f_* is a θ -convergence quotient map. ■

The final proposition follows immediately from Proposition 5.6.

PROPOSITION 6.8. If f is θ -continuous and closure preserving, then f_* is θ -continuous.

We conclude by citing, without detail, some examples which place limitations on the types of results obtained in this section. The function f constructed in Example 4.3 of [14] is perfect but not θ -proper; it is also not difficult to show that in this case f_* is not θ -proper. Thus, in Corollary 6.6, one cannot drop the assumption that f is open. There are other examples which show that f_* may fail to be continuous when f is an open, convergence quotient map, and that f_* may fail to be open when f is open and f_* is continuous.

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