A GENERALIZATION OF CONTRACTION PRINCIPLE

K.M. GHOSH

Dept. of Pure Mathematics
Calcutta University
35, Ballygunge Circular Road
Calcutta - 700019
INDIA

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ABSTRACT: In this paper, a generalized mean value contraction is introduced. This contraction is an extension of the contractions of earlier researchers and of the generalized mean value non-expansive mapping. Using the generalized mean value contraction, some fixed point theorems are discussed.

KEY WORDS AND PHRASES: Fixed Point, Mean Value Iteration.

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1. INTRODUCTION.

Let T be a self mapping of a Banach space E. The mapping T will be called a generalized mean value contraction mapping if for any $x, y \in E$, there exist non-negative real numbers $a_i \ (i = 1, 2, \ldots, 5)$ such that

$$||T\lambda x - T\lambda y|| \leq a_1 ||x - y|| + a_2 ||x - TT\lambda x|| + a_3 ||y - TT\lambda y|| + a_4 ||x - TT\lambda y|| + a_5 ||y - TT\lambda x||$$

(1.1)

where $\sum_{i=1}^{5} a_i < 1$ and $T\lambda x = \lambda x + (1-\lambda) Tx$, and $TT\lambda x = T(\lambda x + (1-\lambda) Tx)$, $0 < \lambda \leq 1$ holds.

The contraction (1.1) is more general than the Banach contraction, contractions of
Kannan [1], Chatterjee [2], Hardy and Rogers [3]. When \( \lambda = 1 \) all these contractions follow as a particular case of (1.1), with suitable choice of \( a_i \)'s. Also, by example, we show that there exist self-mappings which satisfy (1.1), but do not satisfy the well-known contraction just mentioned.

**EXAMPLE 1.** Let \( T \) be a self-mapping on \([0,1]\) defined by

\[
T(0) = 1, \quad T(1) = 0, \quad T(x) = \frac{1}{9}, \quad x \in (0,1).
\]

**EXAMPLE 2.** Let \( T \) be a self-mapping on \([0,1]\) defined by \( T(x) = 1-x, \ x \in [0,1] \).

**EXAMPLE 3.** Let \( T \) be a self-mapping on \([-1,1]\) defined by \( T(x) = -x, \ x \in [-1,1] \).

The mapping \( T \) of the above examples satisfies (1.1) for \( \lambda = \frac{1}{5} \). However, for \( x=0, \ y=1, \) \( T \) of Example 1 or Example 2, and for \( x=1, \ y=-1, \) \( T \) of Example 3 do not satisfy the above well-known contractions. Next, we define generalized mean value non-expansive mapping: Let \( T \) be a self-mapping of a Banach space \( E \). Then \( T \) will be called a generalized mean value non-expansive mapping if for any \( x, y \in E \), there exists non-negative real numbers \( a_i \ (i = 1, 2, \ldots, 5) \) such that

\[
\left| TT_{\lambda} x - TT_{\lambda} y \right| \leq a_1 \left| x - y \right| + a_2 \left| x - TT_{\lambda} x \right| + a_3 \left| y - TT_{\lambda} y \right| + a_4 \left| x - TT_{\lambda} y \right| + a_5 \left| y - TT_{\lambda} x \right|,
\]

where \( \sum_{i=1}^{5} a_i = 1 \) and \( TT_{\lambda} x = \lambda x + (1-\lambda) \ Tx, \ 0 < \lambda \leq 1 \) holds.

Now we define a new contraction which is more general than (1.1) as follows:

Let \( X \) be subset of a normed linear space \( E \). A mapping \( T: X \rightarrow X \) is called an iteratively mean value contraction mapping if for every \( x \in X \) there exist non-negative real numbers \( a \), such that

\[
\left| TT_{\lambda} (TT_{\lambda} x) - TT_{\lambda} x \right| \leq a \left| TT_{\lambda} x - x \right|,
\]

where \( 0 < \lambda \leq 1 \) and \( TT_{\lambda} x = \lambda x + (1-\lambda) \ Tx \) and \( TT_{\lambda} x = T (\lambda x + (1-\lambda) T x) \) holds.

The above definition is given because there are self-mappings of a subset of a normed linear space, which do not satisfy (1.1), but satisfies (1.3). An example of self-mapping for which (1.3) holds but (1.1) does not hold, is given below:

**EXAMPLE 4.** Let \( T \) be a self-mapping on \([-1,7]\) defined by

\[
Tx = -x, \ x \in [-1,1], \ Tx = \frac{6}{7} -x, \ x \in [1,7].
\]
2. MAIN THEOREMS.

THEOREM 1. Let \( T \) be a self-mapping of a normed linear space \( E \). If

(i) \( T \) satisfies (1.1),

(ii) \( \{x_n\} \) converges to \( u \in E \) where \( x_n = TT\lambda x_{n-1} \) (\( n=1,2,\ldots \)) for any \( x_0 \in E \),

(iii) \( T(\lambda u + (1-\lambda) Tu) = \lambda Tu + (1-\lambda) T^2u \), only for \( u \);

then \( T \) has a unique fixed point in \( E \).

PROOF: Let \( x_0 \) be any point in \( E \). Define, \( x_n = TT\lambda x_{n-1} \) (\( n = 1,2,\ldots \)). Put \( x_0 = x \) and \( x_1 = y \) in (1.1), then we have

\[
\|x_1 - x_2\| \leq a_1 \|x_0 - x_1\| + a_2 \|x_0 - x_1\| + a_3 \|x_1 - x_2\| + a_4 \|x_0 - x_2\|, \quad (2.1)
\]

Again, put \( x_1 = x \) and \( y = x_0 \) in (1.1). Then

\[
\|x_2 - x_1\| \leq a_1 \|x_1 - x_0\| + a_2 \|x_1 - x_2\| + a_3 \|x_0 - x_1\| + a_5 \|x_0 - x_2\|. \quad (2.2)
\]

Adding (2.1) and (2.2), we obtain

\[
\|x_n - x_{n+1}\| \leq r \|x_1 - x_0\|,
\]

where \( r = \frac{2a_1 + a_2 + a_3 + a_4 + a_5}{2 - a_2 - a_3 - a_4 - a_5} \) and \( r < 1 \), since \( \sum_{i=1}^{5} a_i < 1 \).

By induction it may be proved that \( \|x_n - x_{n+1}\| \leq r^n \|x_1 - x_0\| \)

It may be shown by routine calculation that \( \{x_n\} \) is a Cauchy sequence. Hence \( \{x_n\} \) is convergent. So, by (ii), \( x_n \to u \in E \), as \( n \to \infty \).

Now, \( \|u - TT\lambda u\| \leq \|u - x_{n+1}\| + \|TT\lambda x_n - TT\lambda u\| \)

\[
\leq \|u - x_{n+1}\| + a_1 \|x_n - u\| + a_2 \|x_{n-1} - x_n\| + a_3 \|u - TT\lambda u\| + a_4 \|x - TT\lambda u\| + a_5 \|u - x_{n+1}\|
\]

\[
\leq (a_3 + a_4) \|u - TT\lambda u\|, \quad \text{as} \; n \to \infty .
\]

Therefore, \( (1 - a_3 - a_4) \|u - TT\lambda u\| \leq 0 \), which implies that \( u = TT\lambda u \), since \( \sum_{i=1}^{5} a_i < 1 \). Now, \( Tu = T(TT\lambda u) = T(T(\lambda u + (1-\lambda) Tu) = T(\lambda Tu + (1-\lambda) T^2u) \), by (i).

Therefore,

\[
\|u - Tu\| = \|T(\lambda u + (1-\lambda) Tu) - T(\lambda Tu + (1-\lambda) T^2u)\| \leq r \|u - Tu\|, \quad \text{by (i)}.
\]

Since \( r < 1 \), \( (1-r) \|u - Tu\| \leq 0 \) implies \( Tu = u \) i.e. \( u \) is a fixed point of \( T \).

Uniqueness of the fixed point follows easily.
THEOREM 2. Let $T$ be a self-mapping of a bounded convex subset $M$ of a normed linear space $E$. If for any $x \in M$,

(i) $T$ satisfies (1.3)

(ii) \{x_n\} converges to $u \in M$, whenever \{x_n\} is convergent, where $x_n = TT_\lambda x_{n-1}$, \((n = 1,2,3,...)\) for any $x_0 \in M$.

(iii) $\lim_{n \to \infty} T(\lambda x_n + (1-\lambda) T x_n) = T(\lambda \lim_{n \to \infty} x_n + (1-\lambda) T \lim_{n \to \infty} x_n)$

(iv) $T(\lambda u + (1-\lambda) Tu) = \lambda Tu + (1-\lambda) T^2 u$, for all $u$;

then $T$ has a fixed point.

PROOF: Proof is exactly similar to that of Theorem 1, so we omit it.

THEOREM 3. Let $E$ be a rotund Banach space, $M$ be a compact convex subset of $E$ and $T$ be a self-mapping of $M$. If $T$ is continuous and $T$ satisfies (1.2) and $TT_\lambda x = T_\lambda Tx$ for any $x \in M$, then $T$ has a fixed point in $M$.

PROOF: Let $x$ be any point in $M$. Define $f(x) = \| x - Tx \|$. Since $T$ and $\| \cdot \|$ are continuous functions, therefore, $f(x)$ is also continuous. So $f(x)$ attains its minimum for some $x$ (say $x = z \in M$).

First suppose $ \| Tz - z \| \neq 0$, then $z$ is a fixed point of $T$. Now let $\| Tz - z \| \neq 0$. Hence

\[
f(T_\lambda z) = \| T_\lambda z - T(T_\lambda z) \| = \| T_\lambda z - T_\lambda (Tz) \| \\
\leq \| z - Tz \| < \| z - Tz \|, \text{ since } E \text{ is rotund.}
\]

$= f(z)$, which contradicts the minimality of $f(z)$.

Therefore $\| T(z) - z \| = 0$ i.e. $Tz = z$ is a fixed point of $T$.

THEOREM 4. Let $E$ be a Banach space, $M$ be a compact convex subset of $E$, and $T$ be a continuous self-mapping of $M$. If for any $x,y \ (x \neq y) \in M$, $T$ satisfies (1.1) (where $\leq$ is replaced by $<$) and $\sum_{i=1}^{5} a_i = 1$ and $T_\lambda x = T_\lambda Tx$, then $T$ has a unique fixed point in $M$.

PROOF: Proof is similar to that of Theorem 3.

3. CONCLUDING REMARKS.

(i) That the condition (iii) of Theorem 1 is necessary for existence of fixed point of $T$ as illustrated by the following example.
EXAMPLE 4. Let $T$ be a self-mapping on $[0,1]$ defined by $T_x = 1 - x$, $x \in [0,1]$, $T(1) = 0$. Here $T$ satisfies conditions (i) and (ii) of Theorem 1 for $\lambda < 1$, but it does not satisfy (iii) and $T$ has no fixed point in $[0,1]$.

(ii) The self-mapping $T$ of Example 1 and Example 2 are non-expansive $(\|Tx - Ty\| \leq \|x - y\|)$. Kirk [4] has proved the following fixed point theorem on non-expansive mapping:

"If $K$ be a nonempty closed convex bounded subset of a reflexive Banach space $X$ and if $K$ possesses normal structure, then every non-expansive mapping from $K$ into itself has a fixed point."

The same result is also established independently by Browder [5] in a uniformly convex Banach space. There is a close connection between the theorems of Kirk and Browder. This was first noted by Goebel [6] that if $X$ be a uniformly convex Banach space, then any closed convex bounded subset $K$ of $X$, must have normal structure.

We observe that for the existence of a fixed point of any non-expansive mapping in a Banach space, the Banach space must have a property either "uniform convexity" or "reflexivity with normal structure". Though self-mapping $T$ in Example 1 and Example 2 are non-expansive, they are contractions in the sense (1.1). These mappings satisfy all the conditions of Theorem 1. Theorem 1 explains the existence of the fixed point of the above mappings without assuming "uniform convexity" or "reflexivity with normal structure".

These examples also suggest that non-expansive mappings may be converted into contraction mappings (general process of conversion is not known). Since the study of contraction mappings is easier than non-expansive mapping, so this type conversion has some importance in fixed point theory.

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