RESEARCH NOTE

DETERMINATION OF THE RESPONSE OF A CLASS OF NONLINEAR TIME INVARIANT SYSTEMS

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ABSTRACT. This paper illustrates by means of a simple example a new approach for the determination of the time domain response of a class of nonlinear systems. The system under investigation is assumed to be described by a nonlinear differential equation with forcing term. The response of the system is first obtained in terms of the input in the form of a Volterra functional expansion. Each of the components in the expansion is first transformed into a multidimensional frequency domain and then to a single dimensional frequency domain by the technique of association of variables. By taking into consideration the conditions for the rapid convergence of the functional expansion the response of the system in the frequency domain can effectively be obtained by taking only the first few terms of the expansion. Time domain response is then found by inverse Laplace transform.

KEY WORDS AND PHRASES. Association of Variables, Functional Expansion, Multidimensional Frequency Domain, Nonlinear Impulse Response, Nonlinear Transfer Function.

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1. INTRODUCTION.

The input/output relationship of a nonlinear system can be viewed with the aid of a "blackbox" as shown in Figure 1. Here x is the input to the system and y is the response, both are functions of time t .

It is the purpose of this paper to determine $y(t)$ when the relationship between $x(t)$ and $y(t)$ is given by

$$L(p)y + \sum_{i=2}^m b_i y^i = x(t), \quad t \geq 0 \quad (1.1)$$

where $L(p) = \sum_{i=0}^n a_i p^i$, and a_i, b_i are constants, $p = \frac{d}{dt}$.

Figure 1 - Input Output Relationship of a Nonlinear System

The response $y(t)$ can be expressed in terms of the input $x(t)$ by the following functional expansion [1] where the limits of integration are from 0 to ∞

$$y(t) = \sum_{n=1}^{\infty} \int \dots \int h_n(\tau_1, \tau_2, \dots, \tau_n) x(t-\tau_1) \dots x(t-\tau_n) d\tau_1 \dots d\tau_n \quad (1.2)$$

As an analogy to the linear system, the n -th order kernel $h_n(\tau_1, \dots, \tau_n)$ can be called a nonlinear impulse response of order n . Let

$$y_n(t) = \int \dots \int h_n(\tau_1, \tau_2, \dots, \tau_n) x(t-\tau_1) \dots x(t-\tau_n) d\tau_1 \dots d\tau_n \quad (1.3)$$

Then

$$y(t) = \sum_{n=1}^{\infty} y_n(t) \quad (1.4)$$

Thus the arbitrary nonlinear system of Figure 1 can be broken down into an infinite number of subsystems connected in parallel as shown in Figure 2.

A significant advantage of the functional expansion of Equation (1.1) is that the analysis of the system can be carried out in the frequency domain by using multidimensional Laplace transform.

2. MULTIDIMENSIONAL FREQUENCY DOMAIN.

Let us introduce the variables $t_1 = t_2 = \dots t_n = t$

Then from (1.3)

$$y_n(t_1, t_2, \dots, t_n) = \int \dots \int h_n(\tau_1, \tau_2, \dots, \tau_n) x(t_1 - \tau_1) \dots x(t_n - \tau_n) d\tau_1 \dots d\tau_n \quad (2.1)$$

Figure 2 - Decomposition of a Nonlinear System into an Infinite Subsystems.

Now by taking n -dimensional Laplace transform [2]

$$y_n(s_1, s_2, \dots, s_n) = H_n(s_1, s_2, \dots, s_n) X(s_1) \dots X(s_n) \quad (2.2)$$

Once the nonlinear transfer function $H_n(s_1, s_2, \dots, s_n)$ is known $Y_n(s_1, s_2, \dots, s_n)$ can again be converted into a single dimensional Laplace transform by the technique called association of variable [3,4]. Now it is a simple matter to determine $y_n(t)$ from $Y_n(s)$ by inverse Laplace transform.

Detailed technique for the determination of $H_n(s_1, s_2, \dots, s_n)$ will not be presented here but can be found in References [5,6]. In general the method is

analogous to the linear system and the first three transfer functions of Equation (1.1) can be shown to have the following expressions.

$$H_1(s_1) = \frac{1}{L(s_1)} \quad (2.3)$$

$$H_2(s_1, s_2) = \frac{-a_2}{L(s_1)L(s_2)L(s_1+s_2)} \quad (2.4)$$

$$H_3(s_1, s_2, s_3) = - \left[\frac{s_2^2}{3L(s_1)L(s_2)L(s_3)} \left\{ \frac{1}{L(s_1+s_2)} + \frac{1}{L(s_2+s_3)} + \frac{1}{L(s_3+s_1)} \right\} \right. \\ \left. + \frac{a_3}{L(s_1)L(s_2)L(s_3)} \right] \frac{1}{L(s_1+s_2+s_3)} \quad (2.5)$$

Let A stand for associate variables $s_1, s_2, \dots, s_n \rightarrow s$. Then from (2.2)

$$Y(s) = \sum_{n=1}^{\infty} AY_n(s_1, s_2, \dots, s_n) \quad (2.6)$$

3. DETERMINATION OF THE TIME DOMAIN RESPONSE.

The evaluation of $y(t)$ from (2.6) will be meaningful only if (2.6) represents a convergent series. The truncation of the series (1.4) at the n -th term will be justified if in the long range contributions by y_{n+1} , y_{n+2} etc., are negligibly small and if so the following approximation will be quite valid

$$y(t) = \sum_{m=1}^n y_m(t) \quad (3.1)$$

One obvious condition for (3.1) is that $|y_1| < \infty$ which implies that the linear component of the system represented by Equation (1.1) is stable. If the steady state values of the components $|y_{nss}| \gg |y_{n+1ss}|$, then y_{n+1} , y_{n+2} etc., will be considered to be negligibly small.

EXAMPLE.

Let a nonlinear system be described by

$$L \frac{di}{dt} + R_1 i + R_2 i^2 = v \quad (3.2)$$

The above system can be identified with a nonlinear RL circuit where v is the voltage step input and the response is the current $i(t)$.

Let $\alpha = R_1/L$, $\beta = R_2/L$, $V = v/L$. From (2.2) and (2.3)

$$I_1(s_1) = \frac{1}{s_1 + \alpha} \cdot \frac{V}{s_1} \quad (3.3)$$

$$i_1(t) = \frac{V}{\alpha} [1 - e^{-\alpha t}]$$

From (2.2) and (2.4)

$$I_2(s_1, s_2) = - \frac{\beta}{(s_1 + \alpha)(s_2 + \alpha)(s_1 + s_2 + \alpha)} \frac{V}{s_1} \frac{V}{s_2}$$

By association of variables

$$I_2(s) = - \frac{2\beta V^2}{s(s+\alpha)^2(s+2\alpha)} \quad (3.4)$$

$$i_2(t) = - \frac{\beta V^2}{\alpha^3} [1 - 2\alpha t e^{-\alpha t} - e^{-2\alpha t}]$$

From (2.2) and (2.5)

$$I_3(s_1, s_2, s_3) = - \frac{\beta^2}{3(s_1+\alpha)(s_2+\alpha)(s_3+\alpha)} \left\{ \frac{1}{s_1+s_2+\alpha} + \frac{1}{s_2+s_3+\alpha} + \frac{1}{s_3+s_1+\alpha} \right\} \\ \cdot \frac{1}{s_1+s_2+s_3+\alpha} \cdot \frac{v^3}{s_1 s_2 s_3}$$

Again by association of variable

$$I_3(s) = - \frac{v^3 \beta^2}{\alpha^2} \left\{ \frac{1}{s(s+\alpha)^2} - \frac{1}{s(s+\alpha)^2(s+2\alpha)^2} \right. \\ \left. - \frac{1}{(s+\alpha)^3(s+2\alpha)} + \frac{1}{(s+\alpha)(s+2\alpha)^2(s+3\alpha)} \right\} \quad (3.5)$$

$$i_{3ss} = - \frac{v^3 \beta^2}{\alpha^5}$$

Now by the application of the final value theorem [7]

$$i_{2ss} = - \frac{BV^2}{\alpha^3} \quad \text{and} \quad i_{3ss} = - \frac{\beta^2 V^3}{\alpha^5}$$

$$i_{3ss}/i_{2ss} = vR_2/R_1^2$$

For any passive circuit it is reasonable to assume $R_1 > R_2$ and for $v < 1$, $i_{3ss} \ll i_{2ss}$. Hence $i_3(t)$ can be considered to be negligible compared to $i_2(t)$. For the above stated conditions the following solution is quite valid.

$$i(t) = \frac{v}{R_1} \left[1 - e^{-\frac{R_1 t}{L}} \right] - \frac{R_2 v}{R_1^3} \left[1 - 2 \frac{R_1}{L} t e^{-\frac{R_1 t}{L}} - e^{-\frac{2R_1 t}{L}} \right] \quad (3.6)$$

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