PARTIAL HENSELIZATIONS

ROBERT W. SHEETS

Department of Mathematics
Southeast Missouri State University
Cape Girardeau, Missouri 63701

(Received January 28, 1980)

ABSTRACT. We define and note some properties of k H-pairs (k Henselian pairs), k N-pairs, and k N'-pairs. It is shown that the 2-Henselization and the 3-Henselization of a pair exist. Characterizations of quasi-local 2H-pairs are given, and an equivalence to the chain conjecture is proved.

KEY WORDS AND PHRASES. k Henselian pair, k N-pair, k N'-pair, chain conjecture.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. 13J15.

1. INTRODUCTION.

We define a pair (A, m) to be a k H-pair (a k Henselian pair) in case the ideal m is contained in the Jacobson radical of the commutative ring A and if for every monic polynomial $f(X)$ of degree k in $A[X]$ such that $\bar{f}(X) \in A/m[X]$ factors into $\bar{f}(X) = \bar{g}_0(X)\bar{h}_0(X)$ where $\bar{g}_0(X)$ and $\bar{h}_0(X)$ are monic and coprime, there exist monic polynomials $g(X), h(X) \in A[X]$ such that $f(X) = g(X)h(X)$, $\bar{g}(X) = \bar{g}_0(X)$, and $\bar{h}(X) = \bar{h}_0(X)$. It is shown that the 2-Henselization and the 3-Henselization of a pair (A, m) exist. Several properties of k H-pairs are noted. And an equivalence to the Chain Conjecture is also given.

2. k H-PAIRS, k N-PAIRS, AND k N'-PAIRS.

In this section we define and give some facts about k H-pairs, k N-pairs, and

k N' -pairs. The main result, Theorem (2.10) states that (i) a k H -pair is a k N -pair, (ii) a k N -pair is a k N' -pair, and (iii) an k N' -pair is a j H -pair provided $k \geq \max \{C_{j,n} \mid n = 0, 1, \dots, j\}$.

We begin by stating several definitions. In these definitions and throughout the paper a ring shall mean a commutative ring with an identity element, and $J(A)$ denotes the Jacobson radical of the ring A .

DEFINITION 2.1. (A, m) is a pair in case A is a ring and m is an ideal in A .

DEFINITION 2.2. (A, m) is a k H -pair in case

(i) $m \subseteq J(A)$; and

(ii) for every monic polynomial $f(X)$ of degree k in $A[X]$ such that $f(X) \in A/m[X]$ factors into $\bar{f}(X) = \bar{g}_0(X) \bar{h}_0(X)$ where $\bar{g}_0(X)$ and $\bar{h}_0(X)$ are monic and coprime, there exist monic polynomials $g(X), h(X) \in A[X]$ such that $f(X) = g(X)h(X)$, $\bar{g}(X) = \bar{g}_0(X)$ and $\bar{h}(X) = \bar{h}_0(X)$.

DEFINITION 2.3. Let (A, m) be a pair. A monic polynomial $X^k + a_{k-1}X^{k-1} + \dots + a_1X + a_0$ of degree k is called a k N -polynomial over (A, m) in case $a_j \in m$ and a_1 is a unit mod m .

DEFINITION 2.4. (A, m) is a k N -pair in case

(i) $m \subseteq J(A)$; and

(ii) every k N -polynomial over (A, m) has a root in m .

The next results give some facts about k N -polynomials and k N -pairs.

LEMMA 2.5. Let $f(X)$ be a k N -polynomial over the pair (A, m) . If $m \subseteq J(A)$, then $f(X)$ has at most one root in m .

PROOF. The proof follows from [5, Lemma 1.5], since a k N -polynomial is an N -polynomial.

REMARK. Every k N -polynomial over a k N -pair (A, m) has one and only one root in m .

PROPOSITION 2.6. If (A, m) is a k N -pair, then (A, m) is an j N -pair for $2 \leq j \leq k$.

PROOF. Given a k N -pair (A, m) , it suffices to show that (A, m) is a $(k-1)$ N -pair. Let $f(X)$ be a $(k-1)$ N -polynomial over (A, m) . Let u be a unit in A and

$g(X) = (X + u)f(X)$. Then $g(X)$ is a k N-polynomial and thus has a root r in m and $0 = g(r) = (r + u)f(r)$. Since $(r + u)$ is a unit, we have $f(r) = 0$. Therefore, (A, m) is a $(k - 1)$ N-pair.

DEFINITION 2.7. Let (A, m) be a pair. A monic polynomial $X^k + d_1 X^{k-1} + d_2 X^{k-2} + \dots + d_k$ of degree k is called a k N'-polynomial over (A, m) in case d_1 is a unit mod m and $d_2, \dots, d_k$ belong to m .

DEFINITION 2.8. (A, m) is a k N'-pair in case

(i) $m \subseteq J(A)$; and

(ii) every k N'-polynomial over (A, m) has a root in A , which is a unit.

We note that if (A, m) is a k N'-pair, $f(X) = X^k + d_1 X^{k-1} + \dots + d_k$ is a k N'-polynomial over (A, m) and $r \in A$ is a root of $f(X)$ given by the definition of a k N'-pair, then $\bar{r} = -\bar{d}_1$, and $f'(r)$ is a unit.

PROPOSITION 2.9. Let (A, m) be a k N'-pair, then (A, m) is an j N'-pair for $2 \leq j \leq k$.

PROOF. Given a k N'-pair (A, m) , it suffices to show that (A, m) is a $(k-1)$ N'-pair. Let $f(X)$ be a $(k-1)$ N'-polynomial over (A, m) . Then $Xf(X)$ is a k N'-polynomial and has a root u , which is a unit. and $uf(u) = 0$ implies that $f(u) = 0$, therefore (A, m) is a $(k-1)$ N'-pair.

THEOREM 2.10. (i) A kH -pair is a kN -pair

(ii) A kN -pair is a kN' -pair

(iii) A kN' -pair is a jH -pair, provided

$$k \geq \max \{C_{j,n} \mid n = 0, 1, \dots, j\}$$

PROOF. Part (i) follows from the definitions.

The proof of (ii) follows from the proof of [10, Lemma 7]

The proof of (iii) follows from Crépeaux's proof of [3, Prop. 1]

3. k N-CLOSURE.

In this section we construct the k N-closure for a given pair (A, m) . That is, we find the "smallest" k N-pair which "contains" (A, m) . The development of this section parallels Greco's development in [5].

In order to construct the k N-closure we need the following definitions.

DEFINITION 3.1. A morphism (of pairs) $\emptyset: (A, m) \rightarrow (B, n)$ is a ring homomorphism $\emptyset: A \rightarrow B$, such that $\emptyset^{-1}(n) = m$.

DEFINITION 3.2. A morphism (of pairs) $\emptyset: (A, m) \rightarrow (B, n)$ is strict in case $n = \emptyset(m)B$ and $\emptyset$ induces an isomorphism $A/m \rightarrow B/n$.

DEFINITION 3.3. Let (A, m) be a pair. A k N -pair (B, n) together with a morphism $\emptyset: (A, m) \rightarrow (B, n)$ is a k N -closure of (A, m) if for any k N -pair (B', n') and any morphism $\Psi: (A, m) \rightarrow (B', n')$ there exists a unique morphism $\Psi': (B, n) \rightarrow (B', n')$ such that $\Psi' \circ \emptyset = \Psi$.

DEFINITION 3.4. Let (A, m) be a pair and $f(X)$ a k N -polynomial over (A, m) . Let $A[x] = A[X]/(f(X))$, $S = 1 + (m, x)A[x]$ and $B = S^{-1}A[x]$. Then (B, mB) is called a simple k N -extension of (A, m) .

DEFINITION 3.5. A k N -extension of (A, m) is a pair obtained from (A, m) by a finite number of simple k N -extensions.

The next two results give some useful properties of simple k N -extensions and k N -extensions.

LEMMA 3.6. Let (B, n) be a simple k N -extension of (A, m) . Let $\emptyset: A \rightarrow B$ be the canonical morphism. Then:

- (i) $x \in n$.
- (ii) $\emptyset^{-1}(n) = m$ and $\emptyset: (A, m) \rightarrow (B, n)$ is a morphism of pairs.
- (iii) $\emptyset: (A, m) \rightarrow (B, n)$ is strict.

PROOF. The proof follows from [5, Lemmas 2.3, 2.4, and 2.5] since a simple k N -extension is a simple N -extension.

COROLLARY 3.7. If (B, n) is a k N -extension of (A, m) , then the canonical morphism $\emptyset: (A, m) \rightarrow (B, n)$ is strict.

We note that a k N -extension of a quasi-local ring (A, m) is a quasi-local ring.

The following lemma is used to show that the partial order defined in Definition (3.9) is well defined.

LEMMA 3.8. Let (A', m') be a k N -extension of (A, m) and let (B, n) be a pair with $n \subseteq J(B)$. Let $\emptyset: (A, m) \rightarrow (A', m')$ be the canonical morphism. Then for any

morphism $\Psi: (A, m) \rightarrow (B, n)$ there is at most one morphism $\Psi': (A', m') \rightarrow (B, n)$ such that $\Psi' \circ \emptyset = \Psi$.

PROOF. The proof follows from [5, Lemma 3.1] since a k N -extension is an N -extension.

In particular, the above lemma holds when (B, n) is a k N -extension of (A, m) .

DEFINITION 3.9. Define a partial order on the set of k N -extensions of (A, m) as follows: If (A', m') and (A'', m'') are two k N -extensions of (A, m) , then $(A', m') \leq (A'', m'')$ if and only if there is a morphism $\Psi: (A', m') \rightarrow (A'', m'')$ such that $\Psi \circ \emptyset = \emptyset''$, where $\emptyset: (A, m) \rightarrow (A', m')$ and $\emptyset'': (A, m) \rightarrow (A'', m'')$ are the canonical morphisms.

PROPOSITION 3.10. Let (A, m) be a pair. Then the k N -extensions of (A, m) form a directed set with the order relation and the morphisms defined above.

PROOF. The proof is analogous to [5, Prop. 3.3].

LEMMA 3.11 Let (A', m') be a k N -extension of (A, m) and let $\emptyset: (A, m) \rightarrow (A', m')$ be the canonical morphism. Let (B, n) be a k N -pair and let $\Psi: (A, m) \rightarrow (B, n)$ be a morphism. Then there is a unique morphism $\Psi': (A', m') \rightarrow (B, n)$ such that $\Psi = \Psi' \circ \emptyset$.

PROOF. The proof is analogous to [5, Prop. 3.4].

THEOREM 3.12. Let (A, m) be a pair and let (A^{kN}, m^{kN}) be the direct limit of the set of all k N -extensions. Then (A^{kN}, m^{kN}) with the canonical morphism $(A, m) \rightarrow (A^{kN}, m^{kN})$ is a k N -closure of (A, m) .

PROOF. The proof is analogous to [5, Thm. 3.5].

We note that if (A, m) is a quasi-local ring; then a k N -closure (A^{kN}, m^{kN}) of (A, m) is quasi-local, since the direct limit of quasi-local rings is quasi-local.

4. k H-CLOSURES AND AN EQUIVALENCE TO THE CHAIN CONJECTURE.

In this section, we note the existence of a $2H$ -closure and of a $3H$ -closure, we give some characterization of a quasi-local $2H$ -pair, and we observe that the H -closure (or Henselization) of a pair (A, m) can be written as the direct limit or union of k H -pairs, $k = 2, 3, 4, \dots$. We also give an equivalence to the Chain Conjecture.

DEFINITION 4.1. Let (A, m) be a pair. A k H -pair (B, n) , together with a

morphism $\emptyset: (A, m) \rightarrow (B, n)$ is a k H-closure of (A, m) if for any k H-pair (B', n') and any morphism $\Psi: (A, m) \rightarrow (B', n')$, there exists a unique morphism $\Psi': (B, n) \rightarrow (B', n')$ such that $\Psi' \circ \emptyset = \Psi$.

THEOREM 4.2. Let (A, m) be a pair. Then:

- (i) a 2 H-closure of (A, m) is (A^{2N}, m^{2N}) .
- (ii) a 3 H-closure of (A, m) is (A^{3N}, m^{3N}) .

PROOF. It suffices to show that a k N-closure ($k = 2, 3$) is a k H-pair. And by Theorem 2.10, we have that a $2N$ -pair is a $2H$ -pair, and that a $3N$ -pair is a $3H$ -pair.

DEFINITION 4.3. If $\emptyset: A \rightarrow B$ is a ring homomorphism, then B is said to be k-integral over A in case each $b \in B$ satisfies a monic polynomial of degree k over $\emptyset(A)$.

REMARK. If B is k -integral over A , then B is also j -integral over A for all $j \geq k$.

In the next three items we give examples of rings and elements which are k -integral over a given ring A .

LEMMA 4.4. If A is an integrally closed domain and $f(X) \in A[X]$ is a monic polynomial of degree k , then $A[X]/(f(X))$ is k -integral over A .

PROOF. Let $A[x] = A[X]/(f(X))$ and let L be the quotient field of A . Then $[L(x):L] \leq k$ and thus each $\alpha \in A[x]$ satisfies a monic polynomial $g(X) \in L[X]$ of degree $\leq k$. Since α is integral over A and A is integrally closed, it follows that $g(X) \in A[X]$. Therefore $A[x]$ is k -integral over A .

LEMMA 4.5. Let A be a ring and let $f(X) = X^2 + \alpha X + \beta \in A[X]$. Then $A[X]/(f(X))$ is 2-integral over A .

PROOF. Let $A[x] = A[X]/(f(X))$ and then all of the elements of $A[x]$ are of the form $ax + b$ where $a, b \in A$. To show that $A[x]$ is 2-integral over A , we need to find $F, G \in A$ such that

$$(ax + b)^2 + F(ax + b) + G = 0.$$

By expanding the left side, we see that $F = a\alpha - 2b$ and $G = a^2\beta - b^2 - Fb = a^2\beta + b^2 - abu$ are the needed values. Therefore $A[X]$ is 2-integral over A .

EXAMPLE 4.6. Each element of $\text{End}_A(A^k)$ is k -integral over A by [1, Proposition 2.4].

In fact, if M is any A -module generated by k elements, each element of $\text{End}_A(M)$ is k -integral over A .

DEFINITION 4.7. (A, m) is a $(\leq k)$ H-pair in case (A, m) is a j H-pair for $2 \leq j \leq k$.

It follows by Theorem 2.10 that if (A, m) is a j N-pair (or j H-pair), then (A, m) is a $(\leq k)$ H-pair provided $j \geq \max \{C_{k,n} \mid n = 0, 1, \dots, k\}$. In particular we have that for $k = 2, 3$, or 4 , a k H-pair is also a $(\leq k)$ H-pair.

LEMMA 4.8. Let (A, m) be a quasi-local domain which is a $(\leq k)$ H-pair. Then every k -integral extension domain of A is quasi-local.

PROOF. The proof is analogous to [6, (30.5)]

DEFINITION 4.9. A ring A is decomposed if A is the product of finitely many quasi local rings.

THEOREM 4.10. Let (A, m) be a quasi local ring. Then the following statements are equivalent.

- (i) Every finite 2-integral A -algebra B is decomposed.
- (ii) Every finite free 2-integral A -algebra B is decomposed.
- (iii) Every A -algebra of the form $A[X]/(f(X))$, where $f(X) \in A[X]$ is monic and of degree 2, is decomposed.
- (iv) (A, m) is a 2 H-pair.

PROOF. (i) $\Rightarrow$ (ii) is clear. (ii) $\Rightarrow$ (iii) is clear by (4.5). The proofs that (iii) $\Rightarrow$ (i) and that (iii) $\Leftrightarrow$ (iv) follow classical lines; for example, see [9, Prop. 5, p.2].

THEOREM 4.11. A quasi local domain (A, m) is a 2H-pair if and only if every 2-integral extension domain A' of A is quasi-local.

PROOF. $(\Rightarrow)$ is true by (4.8).

$(\Leftarrow)$. We will show that (A, m) is a 2H-pair by showing that every finite free 2-integral A -algebra is decomposed. Let B be a finite free 2-integral A -algebra. Since B is decomposed if and only if $B/\text{nil rad } B$ is decomposed, we may assume that B is reduced. Since B is flat over A , regular elements of A are also regular in B . Thus the minimal primes of B contract to $\{0\}$ in A . Let $\{P_i\}_{i \in I}$ be the minimal primes of B . Then for each $i \in I$, B/P_i is a 2-integral extension domain of A and is quasi local by the hypothesis. Thus each minimal prime P_i is contained in a unique maximal

ideal. By [2, Proposition 3, p. 329], the set of minimal primes of B is finite.

Let $I_j = \cap_{P_i \subseteq M_j} P_i$ where $M_j, j=1, \dots, n$, are the maximal ideals of B . Then the

I_j are coprime, and $\cap_{j=1}^n I_j = 0$ since B is reduced. So by the Chinese Remainder Theorem $B \cong \prod_{j=1}^n B/I_j$ and each B/I_j is quasi local. Thus B is decomposed and therefore (A, m) is a 2H-pair.

COROLLARY 4.12. Let (A, m) be a quasi local domain which is 2H-pair. Let A' be an integral extension domain of A . If $b \in A'$ is 2-integral over A , then $b \in J(A')$ or b is a unit.

PROOF. $A[b]$ is a 2-integral extension domain of A and is thus quasi local. The result follows since all the maximal ideals of A' contract to the unique maximal ideal of $A[b]$.

We will now show that the N -closure of a pair (A, m) is the direct limit of the k N -closures of (A, m) . It will follow from this result that the H -closure of (A, m) can be written as the direct limit of k H -pairs.

DEFINITION 4.13. Let (A, m) be a pair. Then (A, m) is an N -pair (respectively, a H -pair) in case (A, m) is a k N -pair (respectively, a k H -pair) for $k = 2, 3, \dots$.

DEFINITION 4.14. Let (A, m) be a pair. An N -pair (respectively, an H -pair) (B, n) , together with a morphism $\emptyset: (A, m) \rightarrow (B, n)$ is an N -closure (respectively, an H -closure) of (A, m) if for any N -pair (respectively, any H -pair) (B', n') , and any morphism $\Psi: (A, m) \rightarrow (B', n')$, there exists a unique morphism $\Psi': (B, n) \rightarrow (B', n')$ such that $\Psi' \circ \emptyset = \Psi$.

THEOREM 4.15. Let (A, m) be a pair. Then the H -closure of (A, m) is isomorphic to the N -closure.

PROOF. See [5, Lemma 1.4 and Theorem 5.10].

PROPOSITION 4.16. Let (A^N, m^N) be an N -closure of (A, m) . Then $(A^N, m^N) \cong \text{dir lim } (A^{kN}, m^{kN})$, where the directed system $\{(A^{kN}, m^{kN}), \mu_{kj}\}$ of k N -closures of (A, m) , $k=2, 3, 4, \dots$, is ordered by $(A^{kN}, m^{kN}) \leq (A^{jN}, m^{jN})$ iff $k \leq j$ and if $k \leq j$, then $\mu_{kj}: (A^{kN}, m^{kN}) \rightarrow (A^{jN}, m^{jN})$ is the unique morphism which makes the following diagram commute:

$$\begin{array}{ccc}
 (A, m) & \xrightarrow[\vartheta_k]{\vartheta} & (A^{kN}, m^{kN}) \\
 & \searrow \vartheta_j & \downarrow u_{kj} \\
 & & (A^{jN}, m^{jN})
 \end{array}$$

where ϑ_j and ϑ_k are the canonical morphisms.

PROOF. The proof follows immediately from Definitions (3.3) and (4.14) and the definition of a direct limit.

COROLLARY 4.17. Let (A^H, m^H) be the H-closure of (A, m) . Then $(A^H, m^H) \cong \text{dir lim } (A_i, m_i)$ where (A_i, m_i) is an i H-pair for $i = 2, 3, \dots$.

PROOF. For a given i , let $(A_i, m_i) = (A^{kN}, m^{kN})$ where $k = \max \{C_{j,n} | n=0, 1, \dots, j\}$. Then the corollary follows by results (2.10), (4.15) and (4.16).

We now give an equivalence to the Chain Conjecture. The terminology used is the same as in [8] or [10].

THEOREM 4.18. The following statements are equivalent:

- (i) The Chain Conjecture holds.
- (ii) Every 2 Henselian local domain A , such that the integral closure of A is quasi-local, is catenary.

PROOF. (i) $\Rightarrow$ (ii). This follows by [8, Thm. 2.4].

(ii) $\Rightarrow$ (i). By [8, Thm. 2.4] it suffices to show that every Henselian local domain is catenary. Let A be a Henselian local domain. Then A is also 2 Henselian and the integral closure of A is quasi-local by [6, (43.12)]. Thus by the hypothesis A is catenary.

5. EXAMPLES.

In this section we show that there exist k N-pairs which are not N-pairs and there exist k H-pairs which are not H-pairs. More precisely, for each prime number p we give an example of a pair which is not a p N-pair but is a k N-pair for $2 \leq k < p$. This example also shows that for any integer $k \geq 2$, there exists a k H-pair which is not a p H-pair for some sufficiently large prime number p .

Let $p > 2$ be a prime number. Let (R, q) be a normal quasi-local domain such that there exists an $f(X) = X^p + \dots + a_1 X + a_0 \in R[X]$, where $a_1 \notin q$, $a_0 \in q$ and $f(X)$

is irreducible over $R[X]$.

In particular, let $R = \mathbb{Z}_{(2)}$ and let $f(X) = X^p + 3X + 6$. Then by Eisenstein's Criterion, $f(X)$ is irreducible in $\mathbb{Q}[X]$, and thus irreducible in $\mathbb{Z}_{(2)}[X]$ since $f(X)$ has content 1.

Let K be the quotient field of R and let $\bar{K}$ be an algebraic closure of K . Let R' be the integral closure of R in $\bar{K}$ and P' any maximal ideal in R' . Now $f(X)$ as an element of $R'[X]$ factors completely, and since $P' \cap R = q$, $f(X)$ has a unique root $\alpha \in P'$. Let L be the least normal extension of K containing α . Then $p \mid [L:K]$ and by [7, Thm. 6] there is a maximal field M without α of exponent p with $K \subseteq M \subseteq \bar{K}$. Let $A = R' \cap M$ and let $m = P' \cap A$.

Now (A, m) is not a p N-pair since $f(X)$ is a p N-polynomial over (A, m) which does not have a root in m . But (A, m) is a k N-pair for $2 \leq k < p$. For, let $g(X)$ be a $(p-1)$ N-polynomial over (A, m) . Then $g(X)$ as an element of $R'[X]$ has a unique root $\beta \in P'$. Now $[M(\beta):M] \leq p-1$, but by [7, Thm. 2], $[M(\beta):M] = p^i$ for some $i \geq 0$. So $[M(\beta):M] = 1$ and $\beta \in M$. Thus $\beta \in m = P' \cap A$ and (A, m) is a $(p-1)$ N-pair. It follows by (2.6) that (A, m) is a k N-pair for $2 \leq k < p$.

REMARK. If j and the prime number p are chosen such that $p > \max \{C_{j,n} \mid n=0,1,\dots,j\}$, then by Theorem 2.10, the above example is an example of a pair (A, m) such that (A, m) is not a p H-pair, but (A, m) is a k H-pair for $2 \leq k \leq j$.

Let the notation be as in the above example. Then (A_m, mA_m) is as an example of a normal quasi-local domain which is not a p N-pair, but is a k N-pair for $2 \leq k < p$.

6. PROPERTIES OF k N-PAIRS.

We conclude this paper by noting that many of the properties of the Hensilization or N-closure of a pair which S. Greco proved in [5] also hold for a k N-closure and thus also for a 2 H-closure and a 3 H-closure. Some of these results are: direct limits commute with k N-closures, cf. [5, Cor. 3.6]; a k N-closure of (A, m) is flat over A and is faithfully flat over A iff $m \subseteq J(A)$, cf. [5, Thm. 6.5]; a k N-closure of a noetherian ring is noetherian, and if a k N-closure of (A, m) is Noetherian and $m \subseteq J(A)$, then A is Noetherian, cf. [5, Cor. 6.9]; if A is Noetherian

and A has one of the properties R_k , S_k , regular, or Cohen-Macaulay, then a k N -closure of (A, m) also has that property, and the converse is also true provided $m \subseteq J(A)$, cf. [5, Cor. 7.7]; a k N -closure preserves locally normal, cf. [5, Thm. 9.7]; and a k N -closure of a reduced ring is reduced, cf. [5, Thm. 8.7].

REFERENCES

1. ATIYAH, M.F. and I.G. MACDONALD. Introduction to Commutative Algebra, Addison-Wesley Publishing Co., Reading, Mass., 1969.
2. BOURBAKI, NICOLAS. Commutative Algebra, Addison-Wesley Publishing Co., Reading, Mass., 1969.
3. CRÉPEAUX, E. "Une caractérisation des couples Henseliens," L'Enseignement Mathématique **13**, (1968), pp. 273-279.
4. GRECO, SILVO. "Algebras over nonlocal Hensel rings," Jour. Algebra **8** (1968), pp. 45-49.
5. GRECO, S. "Henselization of a ring with respect to an ideal," Trans. Amer. Math. Soc. **144** (1969), pp. 43-65.
6. NAGATA, MASAYOSHI. Local Rings, Interscience Publishers, New York, N.Y., 1962.
7. QUIGLEY, FRANK. "Maximal subfields of an algebraically closed field not containing a given element," Proc. Amer. Math. Soc. **13** (1962), pp. 562-566.
8. RATLIFF, L.J., JR. Chain Conjectures and H-Domains, Lecture Notes in Mathematics 311, Springer-Verlag, New York, N.Y., 1973, pp. 222-238.
9. RAYNAUD, MICHEL. Anneaux Locaux Henseliens, Lecture Notes in Mathematics 169, Springer-Verlag, New York, N.Y., 1970.
10. SCHERZLER, EBERHAND. "On Henselian Pairs," Commu. Algebra **3** (1975), pp. 391-404.