ON K-TRANSFORM

C. NASIM

Department of Mathematics
The University of Calgary
Calgary, Alberta
Canada

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ABSTRACT.

Using a combination of infinite order linear differential operators and integral operators, the inversion of K -transform is established. Inversion procedures for Laplace transform and Potential transform are derived as special cases.

KEY WORDS AND PHRASES. K -transform, Laplace transform, differential operators.

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1. INTRODUCTION.

In this paper, we discuss the inversion of the K -transform, in Hilbert space $L^2(0, \infty)$. The method involves only the real analysis and employs differential operators of infinite order, cf [4] and [6, Chapter vii]. An algorithm for

inverting the bilateral Laplace transform is established as a special case. Also some examples are given to illustrate the procedure.

LEMMA 1. [5, p.94] Let $f(x) \in L^2(0, \infty)$ and

$$F(s) = \int_0^{\infty} f(x)x^{s-1}dx, \quad s = \frac{1}{2} + it, \quad -\infty < t < \infty,$$

then

$$f(x) = \frac{1}{2\pi i} \lim_{t \rightarrow \infty} \int_{\frac{1}{2}-it}^{\frac{1}{2}+it} F(s)x^{-s}ds.$$

F is called the Mellin transform of f .

LEMMA 2. [5, p.95] If f and $g \in L^2(0, \infty)$ and have Mellin transforms F and G respectively, then

$$\int_0^{\infty} \frac{1}{x} f\left(\frac{1}{x}\right) g(xt)x^{s-1}dx = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} F(s)G(s)t^{-s}ds.$$

Next some operational considerations.

If $\theta = -x \frac{d}{dx}$, then it is an easy matter to see that

$$\theta^n [x^{-s}] = s^n x^{-s}, \quad s = \sigma + it, \quad -\infty < t < \infty.$$

Therefore

$$p_n(\theta) [x^{-s}] = p_n(s) x^{-s},$$

where $p_n(\theta)$ is a polynomial of degree n in θ ; consequently

$$p(\theta) [x^{-s}] = \lim_{n \rightarrow \infty} p_n(\theta) [x^{-s}]$$

$$= p(s) x^{-s}.$$

Also,

$$\begin{aligned} n^\theta [x^{-s}] &= e^{\theta \ln n} [x^{-s}] \\ &= \lim_{N \rightarrow \infty} \sum_{k=0}^N \frac{(\theta \ln n)^k}{k!} [x^{-s}] \\ &= \lim_{N \rightarrow \infty} \sum_{k=0}^N \frac{(s \ln n)^k}{k!} x^{-s} = n^s x^{-s}. \end{aligned}$$

With this understanding, we interpret the operator $\frac{1}{\Gamma(1-\theta)}$, a linear differential operator of infinite order, [6, p.234], such that

$$\begin{aligned} \frac{1}{\Gamma(1-\theta)} [x^{-s}] &= \lim_{n \rightarrow \infty} n^\theta \prod_{k=1}^n \left(1 - \frac{\theta}{k}\right) [x^{-s}] \\ &= \frac{1}{\Gamma(1-s)} x^{-s}. \end{aligned}$$

Similarly

$$\frac{1}{\Gamma(\alpha - \beta\theta)} [x^{-s}] = \frac{1}{\Gamma(\alpha - \beta s)} x^{-s}. \quad (1.1)$$

Next we define the operator $\Gamma(\alpha - \beta\theta)$ as having the property that

$$\Gamma(\alpha - \beta\theta)[x^{-s}] = \Gamma(\alpha - \beta s)x^{-s}. \quad (1.2)$$

This is not a differential operator in the above sense, but behaves in the manner of (1.2) for all $s = \sigma + it$, $-\infty < t < \infty$, $R(s) < \frac{\alpha}{\beta}$.

2. THE MAIN RESULT

THEOREM 1. Let the functions f and $\phi \in L^2(0, \infty)$. Define

$$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty (xt)^\alpha K_\mu(xt) \phi(t) dt,$$

where $|\mu| \leq \frac{1}{2}$, $\alpha > 0$ and K_μ being the usual modified Bessel function of order μ .

If

$$R(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\infty} (xt)^{\beta} J_{\nu}(xt) f(t) dt ,$$

$|\nu| \leq \frac{1}{2}$, $\beta > 0$ then

$$\frac{\pi 2^{-(\alpha+\beta-1)} \Gamma\left[\frac{1}{2}(\nu-\beta+2-\theta)\right]}{\Gamma\left[\frac{1}{2}(\mu+\alpha+1-\theta)\right] \Gamma\left[\frac{1}{2}(\nu+\beta+\theta)\right] \Gamma\left[\frac{1}{2}(\alpha-\mu+1-\theta)\right]} [R(x)] = \phi(x), \quad \text{a.a. } x > 0.$$

PROOF. Note that the integral defining the function f is absolutely convergent, since $\phi \in L^2(0, \infty)$ and $t^{\alpha} K_{\mu}(t) \in L^2(0, \infty)$ due to the hypotheses.

Now,

$$\begin{aligned} R(x) &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\infty} (xt)^{\beta} J_{\nu}(xt) f(t) dt \\ &= \frac{2}{\pi} \int_0^{\infty} (xt)^{\beta} J_{\nu}(xt) dt \int_0^{\infty} (ut)^{\alpha} K_{\mu}(ut) \phi(u) du \\ &= \frac{2}{\pi} x^{\beta} \int_0^{\infty} u^{\alpha} \phi(u) du \int_0^{\infty} t^{\alpha+\beta} J_{\nu}(xt) K_{\mu}(ut) dt . \end{aligned} \quad (2.1)$$

The change of order of integration is justified by absolute convergence

$$\int_0^{\infty} \int_0^{\infty} |t^{\alpha+\beta} u^{\alpha} J_{\nu}(xt) K_{\mu}(ut) \phi(u)| du dt < \infty$$

using the asymptotic expansions of the Bessel functions J and K . The t -integral can be evaluated, [1(II), p.137] so that (2.1) above gives,

$$R(x) = \frac{2^{\alpha+\beta}}{\pi} \frac{\Gamma(a)\Gamma(b)}{\Gamma(\nu+1)} \int_0^{\infty} x^{\beta+\nu} u^{-(\nu+\beta+1)} {}_2F_1\left(a, b; \nu+1; \frac{-x^2}{u^2}\right) \phi(u) du ,$$

where $R(\nu+1+\alpha+\beta) > |R(\mu)|$ and $a = \frac{1}{2}(\nu+\mu+1+\alpha+\beta)$, $b = \frac{1}{2}(\nu-\mu+1+\alpha+\beta)$, ${}_2F_1$ being the Hypergeometric function.

Let,

$$R(x) = \int_0^\infty \frac{1}{u} \phi\left(\frac{1}{u}\right) k(xu) du,$$

where

$$k(x) = \frac{2^{\alpha+\beta}}{\pi} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\nu+1)} x^{\alpha+\beta} {}_2F_1(\alpha, \beta; \nu+1; -x^2).$$

Now $k \in L^2(0, \infty)$ since $\beta + \nu > -\frac{1}{2}$ and $|\mu| < \alpha + \frac{1}{2}$, and further $\frac{1}{x} \phi\left(\frac{1}{x}\right) \in L^2(0, \infty)$, since ϕ does, hence by Lemma 2, we have

$$R(x) = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Phi(s) K(s) x^{-s} ds,$$

where Φ and K are the Mellin transforms of ϕ and k respectively and [1(I), p.336],

$$K(s) = \frac{2^{\alpha+\beta-1}}{\pi} \frac{\Gamma\left[\frac{1}{2}(\mu+\alpha+1-s)\right] \Gamma\left[\frac{1}{2}(\nu+\beta+s)\right] \Gamma\left[\frac{1}{2}(\alpha-\mu+1-s)\right]}{\Gamma\left[\frac{1}{2}(\nu-\beta+2-s)\right]}$$

Next, we apply the operator $\frac{1}{K(\theta)}$, where $\theta = -x \frac{d}{dx}$ to the function $R(x)$, to obtain

$$\begin{aligned} \frac{1}{K(\theta)} [R(x)] &= \frac{1}{K(\theta)} \left[\frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Phi(s) K(s) x^{-s} ds \right] \\ &= \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Phi(s) K(s) ds \frac{1}{K(\theta)} [x^{-s}]. \end{aligned}$$

Using the results (1.1) and (1.2), we see that $\frac{1}{K(\theta)} [x^{-s}] = \frac{1}{K(s)} x^{-s}$.

Thus,

$$\begin{aligned} \frac{1}{K(\theta)} [R(x)] &= \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Phi(s) K(s) \frac{x^{-s}}{K(s)} ds \\ &= \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Phi(s) x^{-s} ds \\ &= \Phi(x), \quad \text{a.a. } x > 0, \end{aligned}$$

as required. The bringing of the operator $\frac{1}{K(\theta)}$ inside the integral, amounts to

nothing more than differentiating inside the integral sign.

EXAMPLE 1. Let $\phi(x) = J_{\eta}(x)$. Then

$$\begin{aligned} f(x) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} (xt)^{\alpha} J_{\eta}(t) K_{\mu}(xt) dt \\ &= \frac{2^{\alpha-\frac{1}{2}}}{\pi^{\frac{1}{2}}} x^{-(\eta+1)} \frac{\Gamma\left[\frac{1}{2}(\eta+\mu+\alpha+1)\right] \Gamma\left[\frac{1}{2}(\eta-\mu+\alpha+1)\right]}{\Gamma(\eta+1)} \\ &\quad \cdot {}_2F_1\left(\frac{1}{2}(\eta+\mu+\alpha+1), \frac{1}{2}(\eta-\mu+\alpha+1); \eta+1; -\frac{1}{x^2}\right) \end{aligned}$$

[1(II), p.137]. And

$$R(x) = \frac{2^{\alpha}}{\pi} \frac{\Gamma(c)\Gamma(d)}{\Gamma(\eta+1)} x^{\beta} \int_0^{\infty} t^{\beta-\eta-1} {}_2F_1\left(c, d; \eta+1; -\frac{1}{t^2}\right) J_{\nu}(xt) dt$$

where $c = \frac{1}{2}(\eta+\mu+\alpha+1)$, $d = \frac{1}{2}(\eta-\mu+\alpha+1)$.

$$= \frac{2^{\alpha+\beta-\eta-1}}{\pi} x^{\eta} G_{24}^{22} \left[\frac{1}{4} x^2 \left| \begin{matrix} 1-c, 1-d \\ \frac{1}{2}(\beta-\eta+\nu), 0, -\eta, \frac{1}{2}(\beta-\eta-\nu) \end{matrix} \right. \right],$$

[1(II), p.82], where G is Meijer's G Function.

Now

$$\begin{aligned} \frac{1}{K(\theta)} R(x) &= \frac{2^{-\eta} \Gamma\left[\frac{1}{2}(\nu-\beta+2-\theta)\right]}{\Gamma\left[\frac{1}{2}(\mu+\alpha+1-\theta)\right] \Gamma\left[\frac{1}{2}(\nu+\beta+\theta)\right] \Gamma\left[\frac{1}{2}(\alpha-\mu+1-\theta)\right]} \\ &\quad \cdot \left[x^{\eta} G_{24}^{22} \left[\frac{1}{4} x^2 \left| \begin{matrix} 1-c, 1-d \\ \frac{1}{2}(\beta-\eta+\nu), 0, -\eta, \frac{1}{2}(\beta-\eta-\nu) \end{matrix} \right. \right] \right] \\ &= 2^{-\eta} x^{\eta} G_{02}^{10} \left[\frac{1}{4} x^2 \left| \begin{matrix} 0, -\eta \end{matrix} \right. \right] \\ &= J_{\eta}(x), \text{ as predicted.} \end{aligned}$$

The operator $\frac{1}{K(\theta)}$ applied on $x^n G_{24}^{22}(\cdot)$, is evaluated by writing $G_{24}^{22}(\cdot)$ in terms of the complex integral and then the differential operator being applied inside the integral to give $x^n G_{02}^{10}(\cdot)$.

3. SPECIAL CASES.

(i) Let $\alpha + \beta - 1 = \mu + \nu$. The inversion of the main theorem be

$$\frac{\pi}{2^{\mu+\nu}} \frac{1}{\Gamma\left[\frac{1}{2}(\mu+\alpha+1-\theta)\right] \Gamma\left[\frac{1}{2}(\beta+\nu+\theta)\right]} R(x) = \phi(x),$$

where

$$R(x) = \Gamma(\mu+\nu+1) \frac{2^{\mu+\nu+1}}{\pi} x^{\beta+\nu} \int_0^\infty \frac{u^{\alpha+\mu}}{(x^2+u^2)^{\mu+\nu+1}} \phi(u) du;$$

giving us an inversion of the generalized Potential transform.

The following special cases provide us with a procedure for inverting Laplace transform, cf. [3] and [6, p.232].

(ii) If $\alpha = \beta = \mu = \frac{1}{2}$ and $\nu = -\frac{1}{2}$, then Theorem 1 gives

THEOREM 2. Let f and $\phi \in L^2(0, \infty)$. Define

$$f(x) = \int_0^\infty e^{-xt} \phi(t) dt,$$

the bilateral Laplace transform of ϕ . If

$$R(x) = \frac{2}{\pi} \int_0^\infty \cos xt f(t) dt,$$

then

$$\sin \frac{1}{2} \pi \theta [R(x)] = \phi(x), \quad \text{a.a. } x > 0.$$

Here the operator $\frac{1}{K(\theta)}$ is reduced to the operator $\sin \frac{1}{2} \pi \theta$. We can interpret this operator by considering the product expansion

$$\sin \frac{1}{2} \pi z = \frac{1}{2} \pi z \lim_{n \rightarrow \infty} \prod_{k=1}^n \left(1 - \frac{z^2}{4k^2} \right).$$

Now if $S_n(\alpha) = \frac{1}{2} \pi \alpha \prod_{k=1}^n \left(1 - \frac{\alpha^2}{4k^2}\right)$, then, for example

$$\sin \frac{1}{2} \pi \theta [x^\alpha] = \lim_{n \rightarrow \infty} S_n(-\alpha) x^\alpha = -\sin \frac{1}{2} \pi \alpha x^\alpha.$$

(iii) Put $\alpha = \mu = \frac{1}{2}$ and $\beta = \nu = \frac{1}{2}$ in Theorem 1, then we have

THEOREM 3. Let f and $\phi \in L^2(0, \infty)$. Define

$$f(x) = \int_0^\infty e^{-xt} \phi(t) dt,$$

the Laplace transform of ϕ . If

$$R(x) = \frac{2}{\pi} \int_0^\infty \sin xt f(t) dt,$$

then

$$\cos \frac{1}{2} \pi \theta [R(x)] = \phi(x), \quad \text{a.a. } x > 0.$$

Here again the differential operator $\frac{1}{K(\theta)}$ of Theorem 1 is reduced to $\cos \frac{1}{2} \pi \theta$, which can be interpreted in a similar way as the operator $\sin \frac{1}{2} \pi \theta$. Also, for instance, $\cos \frac{1}{2} \pi \theta [x^\alpha] = \cos \frac{1}{2} \pi \alpha x^\alpha$.

(iv) Hamburger's formula.

EXAMPLE 2. Let $\phi(x) = \sin^{2n} x$. Then by Theorem 3,

$$f(x) = \int_0^\infty \sin^{2n} t e^{-xt} dt = \frac{(2n)!}{x(x^2+2^2)(x^2+4^2)\dots(x^2+(2n)^2)},$$

(Hamburger's formula).

Now

$$\begin{aligned} R(x) &= \frac{2(2n)!}{\pi} \int_0^\infty \frac{\sin xt}{t(t^2+2^2)(t^2+4^2)\dots(t^2+4n^2)} dt \\ &= (-1)^n 2^{-2n} \left[2 \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} e^{2(k-n)x} + (-1)^n \binom{2n}{n} \right], \end{aligned}$$

[2, p.414].

Thus, according to Theorem 3

$$\cos \frac{1}{2} \pi \theta [R(x)] = (-1)^n 2^{-2n} \left[2 \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \cos \frac{1}{2} \pi \theta [e^{2(k-n)x}] + (-1)^n \binom{2n}{n} \right].$$

Now,

$$\begin{aligned} \cos \frac{1}{2} \pi \theta [e^{mx}] &= \cos \frac{1}{2} \pi \theta \sum_{k=0}^{\infty} \frac{(mx)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{m^k}{k!} \cos \frac{1}{2} \pi \theta [x^k] \\ &= \sum_{k=0}^{\infty} \frac{m^k}{k!} \cos \frac{1}{2} \pi k x^k = \sum_{k=0}^{\infty} \frac{m^{2k} (-1)^k n^{2k}}{(2k)!} \\ &= \cos mx. \end{aligned}$$

Hence

$$\begin{aligned} \cos \frac{1}{2} \pi \theta [R(x)] &= (-1)^n 2^{-2n} \left[2 \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \cos [2(k-n)x] + (-1)^n \binom{2n}{n} \right] \\ &= \sin^{2n} x, \end{aligned}$$

[2, p.25], as predicted, by the theorem, verifying the Hamburger formula.

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