A RESULT ON CO-CHROMATIC GRAPHS

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ABSTRACT. A sufficient condition for two graphs with the same number of nodes to have the same chromatic polynomial is given.

KEY WORDS AND PHRASES. Graph, Co-chromatic, Chromatic polynomial.


1. INTRODUCTION.

We prove a theorem which gives a sufficient condition for two graphs to be co-chromatic i.e. to have the same chromatic polynomial.

The chromatic polynomial $\chi(G;\lambda)$ of a graph $G$ with $p$ nodes has degree $p$ and constant term equal to 0. Hence the chromatic polynomial has $p$ coefficients.

If the graph has at least one edge, then the sum of these coefficients is equal to 0. Hence the chromatic polynomial is uniquely determined if $p-1$ of the coefficients are known. Our result is a generalization of this.

2. MAIN RESULTS.
THEOREM 2. If two graphs with \( p \) nodes have chromatic numbers \( \geq n \) and have at least \( p+1-n \) corresponding coefficients of their chromatic polynomials equal, then they are co-chromatic.

In the proof we shall use a special case of the following Lemma.

**LEMMA 2.1**
Let \( P(x) = c_1 x^{n_1} + c_2 x^{n_2} + \ldots + c_s x^{n_s} \), where \( c_i \) and \( n_i \) are real numbers for \( i = 1, 2, \ldots, s \). We assume that \( c_i \neq 0 \) for all \( i \) and that \( n_i \neq n_j \) for \( i \neq j \). Then the equation \( P(x) = 0 \) has at most \( s-1 \) real positive solutions.

**PROOF OF THE LEMMA 2.1** By induction over \( s \). For \( s = 1 \) the statement is obvious. Suppose that it is true for \( s-1 \) and let \( P(x) \) be the above expression. The expression

\[
Q(x) = x^{n_1} \cdot P(x) = c_1 x^{n_2-n_1} + \ldots + c_s x^{n_s-n_1}
\]

has the derivative

\[
Q'(x) = c_2 (n_2-n_1) x^{n_2-n_1-1} + \ldots + c_s (n_s-n_1) x^{n_s-n_1-1}.
\]

By induction, \( Q'(x) = 0 \) for at most \( s-2 \) positive \( x \). But between any two positive solutions of \( P(x) = x^{n_1} \cdot Q(x) = 0 \), there exists a solution of \( Q'(x) = 0 \). Hence \( P(x) = 0 \) for at most \( s-1 \) positive \( x \).

Q.E.D.

**PROOF OF THE THEOREM 2.** Let \( G \) and \( H \) be the two graphs. Let us assume that \( m \) of the coefficients of \( \chi(G;\lambda) \) and \( \chi(H;\lambda) \) are equal. Then \( m \geq p + 1 - n \), by our assumption. Let us assume that \( m < p \). We can therefore write

\[
\chi(G;\lambda) = f(\lambda) + g(\lambda)
\]
and

\[
\chi(H;\lambda) = f(\lambda) + h(\lambda),
\]
where \( f(\lambda) \) contains \( m \) terms and \( g(\lambda) \) and \( h(\lambda) \) are the remaining terms of \( \chi(G;\lambda) \) and \( \chi(H;\lambda) \) respectively.

Since \( G \) and \( H \) have chromatic numbers \( \geq n \), all integers \( 1, 2, \ldots, n-1 \) are roots of \( \chi(G;\lambda) \) and \( \chi(H;\lambda) \).
RESULT ON CO-CHROMATIC GRAPHS

Let
\[ g(\lambda) = a_1\lambda^{n_1} + a_2\lambda^{n_2} + \ldots + a_{p-m}\lambda^{n_{p-m}} \]

and
\[ h(\lambda) = b_1\lambda^{n_1} + b_2\lambda^{n_2} + \ldots + b_{p-m}\lambda^{n_{p-m}} \]

If \( r \) is an integer such that \( 1 \leq r \leq n-1 \), then \( g(r) = h(r) \), i.e.
\[ (a_1-b_1)r^{n_1} + (a_2-b_2)r^{n_2} + \ldots + (a_{p-m}-b_{p-m})r^{n_{p-m}} = 0. \]

Since \( n-1 \geq p-m \), this is a contradiction by the Lemma.

Q.E.D.

3. ILLUSTRATIONS

We will now illustrate the theorem. We will assume that the chromatic polynomial of a graph \( G \) with \( p \) nodes is written in descending powers of \( \lambda \).

i.e.

\[ \chi(G;\lambda) = \sum_{k=0}^{p} a_{p-k}\lambda^{p-k}. \]

It is well known that if \( G \) contains \( p \) nodes and \( q \) edges, then \( a_p, a_{p-1} \) and \( a_{p-2} \) are \( 1, -q \) and \( \binom{q}{2} - A \) respectively, where \( A \) is the number of triangles in \( G \). It was also shown in Farrell [1] (Theorem 1) that

\[ a_{p-2} = -\binom{q}{3} + (q-2)A + B - 2C, \]

where \( B \) and \( C \) are the numbers of subgraphs of \( G \) which are quadrilaterals (without diagonals) and complete graphs with four nodes.

Let \( G_1 \) and \( G_2 \) be the graphs shown below
Let \( \chi(G_1; \lambda) = \sum_{k=0}^{\infty} a_{6-k} \lambda^{6-k} \) and \( \chi(G_2; \lambda) = \sum_{k=0}^{\infty} b_{6-k} \lambda^{6-k} \). Then \( a_6 = b_6 = 1 \) and \( a_5 = b_5 = 11 \). Since \( G_1 \) and \( G_2 \) contain 7 triangles, \( a_4 = b_4 = \binom{11}{2} - 7 = 48 \). Therefore \( G_1 \) and \( G_2 \) have 6 nodes, their chromatic number is \( \geq 4 = n \) and \( (p+1-n) = 3 \) of their corresponding coefficients are equal. It follows from the above theorem that \( G_1 \) and \( G_2 \) are co-chromatic.

The chromatic polynomial of \( G_1 \) and \( G_2 \) has been computed. It is

\[
\chi(G_1; \lambda) = \chi(G_2; \lambda) = \lambda^6 - 11\lambda^5 + 48\lambda^4 - 103\lambda^3 + 107\lambda^2 - 42\lambda.
\]

Consider the graphs \( H_1, H_2 \) and \( H_3 \) shown below.

All three graphs contain 8 nodes and 18 edges. Each contains 17 triangles.

Therefore, the third coefficient of their chromatic polynomial is \( \binom{18}{2} - 17 = 136 \).

Finally, each contains 7 subgraphs which are complete graphs with 4 nodes and 0 quadrilaterals without diagonals. Therefore the fourth coefficients are equal.
Hence from the above theorem $H_1$, $H_2$ and $H_3$ are co-chromatic.

The chromatic polynomial of $H_1$, $H_2$ and $H_3$ has been computed. It is

$$\chi(H_1; \lambda) = \chi(H_2; \lambda) = \chi(H_3; \lambda) = \lambda^8 - 18\lambda^7 + 136\lambda^6 - 558\lambda^5 + 1339\lambda^4 - 1872\lambda^3 + 1404\lambda^2 - 432\lambda.$$

REFERENCES

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