SEMI SEPARATION AXIOMS AND HYPERSPACES

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ABSTRACT. In this paper examples are given to show that s -regular and s -normal are independent; that s -normal, and s -regular are not semi topological properties; and that $(S(X), E(X))$ need not be $\text{semi-}T_1$ even if (X, T) is compact, s -normal, s -regular, $\text{semi-}T_2$, and T_0 . Also, it is shown that for each space (X, T) , $(S(X), E(X))$, $(S(X_0), E(X_0))$, and $(S(X_{S_0}), E(X_{S_0}))$ are homeomorphic, where $(X_0, Q(X_0))$ is the T_0 -identification space of (X, T) and $(X_{S_0}, Q(X_{S_0}))$ is the $\text{semi-}T_0$ -identification space of (X, T) , and that if (X, T) is s -regular and R_0 , then $(S(X), E(X))$ is $\text{semi-}T_2$.

KEY WORDS AND PHRASES. *Semi open sets, semi topological properties, and hyperspaces.*

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1. INTRODUCTION.

Semi open sets were first defined and investigated by Levine [1] in 1963.

DEFINITION 1.1. Let (X, T) be a space and let $A \subseteq X$. Then A is semi open, denoted by $A \in SO(X, T)$, iff there exists $U \in T$ such that $U \subset A \subset \bar{U}$.

Since 1963 semi open sets have been used to define and investigate many new topological properties. Maheshwari and Prasad [2], [3], and [4] generalized T_i , $i = 0, 1, 2$, regular, and normal to $\text{semi-}T_i$, $i = 0, 1, 2$, s -regular, and s -normal,

by replacing the word open in the definitions of T_i , $i = 0, 1, 2$, regular, and normal by semi open, respectively. Except for s-normal and s-regular, the relationships between these separation axioms have been determined. In this paper, the relationship between s-normal and s-regular is determined, and semi topological properties and hyperspaces are further investigated.

2. s-REGULAR, s-NORMAL, AND SEMI TOPOLOGICAL PROPERTIES.

Maheshwari and Prasad [4] gave an example showing that s-normal does not imply s-regular. That example can be combined with the following example to show that s-regular and s-normal are independent.

EXAMPLE 2.1 Let N denote the natural numbers, let T be the discrete topology on N , let e be the embedding map of (N, T) into $\pi\{I_f \mid f \in C^*(N, T)\}$, and let $(\beta N, W) = (\overline{e(N)}, e)$ denote the Stone-Čech compactification of (N, T) . From Willard's book [5], $(\beta N, W)$ is extremely disconnected, $e(N)$ is open in βN , and $B = \beta N - e(N)$ is infinite. For each $p \in N$ let $N_p = \{n \in N \mid n \leq p\}$. Since for each $p \in N$, there exists a function $f_p: N_p \rightarrow B \times W$ such that (1) if $i \in \{2, \dots, p\}$, then f_i is an extension of f_{i-1} , (2) $x_i \in 0_i$ for all $i \in N_p$, (3) if $i, j \in N_p$, then $\overline{0_i} \cap \overline{0_j} \neq \emptyset$ iff $i = j$, and (4) $B - \bigcup_{i=1}^p \overline{0_i}$ is infinite, then there exists a sequence $\{(x_n, 0_n)\}_{n \in N} \subset B \times W$ such that $x_n \in 0_n$ for all $n \in N$ and $\overline{0_m} \cap \overline{0_n} \neq \emptyset$ iff $m = n$. Let $\{a_n\}_{n \in N}$ be a sequence such that $\{a_n \mid n \in N\} \cap \beta N = \emptyset$ and $a_n = a_m$ iff $n = m$, let $V = \{x_n \mid n \in N\} \cup \left(\bigcup_{n \in N} \{U_n = 0_n \cap e(N)\} \right) \subset \beta N$, let W_1 be the relative topology on V , and let $X = V \cup \{a_n \mid n \in N\}$. Since U_1 is countably infinite, then $U_1 = \{y_n\}_{n \in N}$, where $y_i = y_j$ iff $i = j$. For each $i \in N$, let $B_i = \{0 \subset X - (\{x_n \mid n \in N\} \cup \{a_n \mid n \neq i\}) \mid 0 \cap U_n \in W_1 \text{ for all } n \in N, x_n \in \overline{0 \cap U_n} \text{ except for finitely many } n \in N, \text{ and } a_i, y_i \in 0\}$, and let $W_2 = \bigcup_{i \in N} B_i$. Then $W_1 \cup W_2$ is a base for a topology S on X , (X, S) is s-regular, semi- T_2 , and T_0 , and (X, S) is not s-normal since $A = \{a_n \mid n \in N\}$ and $C = \{x_n \mid n \in N\}$ are disjoint closed sets and there do not exist disjoint semi open sets containing A and C , respectively.

Semihomomorphisms and semi topological properties were first introduced and investigated by Crossley and Hildebrand [6].

DEFINITION 2.1. A 1-1 function from one space onto another space is a semihomomorphism iff images of semi open sets are semi open and inverses of semi open sets are semi open. A property of topological spaces preserved by semihomomorphisms is called a semi topological property.

Example 1.5 in [6], which was used to show that normal and regular are not semi topological properties, also shows that s-normal and s-regular are not semi topological properties.

Clearly, semi- T_i , $i = 0, 1, 2$, are semi topological properties.

3. HYPERSPACES AND SEMI SEPARATION AXIOMS

DEFINITION 3.1. Let (X, T) be a topological space, let $A \subset X$, and define $S(X)$, $S(A)$, and $I(A)$ as follows: $S(X) = \{F \subset X \mid F \text{ is nonempty and closed}\}$, $S(A) = \{F \in S(X) \mid F \subset A\}$, and $I(A) = \{F \in S(X) \mid F \cap A \neq \emptyset\}$. Denote by $E(X)$ the smallest topology on $S(X)$ satisfying the conditions that if $G \in T$, then $S(G) \in E(X)$ and $I(G) \in E(X)$. Then $(S(X), E(X))$ is called a hyperspace [7].

Michael [8] showed that for a space (X, T) , $\mathcal{B} = \{ \langle G_1, \dots, G_p \rangle \mid p \in \mathbb{N} \text{ and } G_i \in T \text{ for all } i \in \mathbb{N}_p = \{1, \dots, p\} \}$ is a base for $E(X)$, where $\mathbb{N}$ is the natural numbers and $\langle G_1, \dots, G_p \rangle = \langle G_1 \rangle_{i=1}^p = \{F \in S(X) \mid F \subset \bigcup_{i=1}^p G_i \text{ and } F \cap G_i \neq \emptyset \text{ for all } i \in \mathbb{N}_p\}$, and observed that for each space (X, T) , $(S(X), E(X))$ is T_0 . Since T_0 implies semi- T_0 , then for each space (X, T) , $(S(X), E(X))$ is semi- T_0 . The following example shows that $(S(X), E(X))$ need not be semi- T_1 even if (X, T) is compact, s-normal, s-regular, semi- T_2 , and T_0 .

EXAMPLE 3.1 Let $X = \{a, b, c, d\}$ and $T = \{X, \emptyset, \{b\}, \{d\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}\}$. Then $(S(X), E(X))$ is not semi- T_1 since $\{a, b, c\}, X \in S(X)$ such that $\{a, b, c\} \neq X$ and there does not exist a semi open set containing $\{a, b, c\}$ and not X .

In Willard's book [5], T_0 -identification spaces are discussed.

DEFINITION 3.2 Let R be the equivalence relation on a space (X, T) defined by xRy iff $\overline{\{x\}} = \overline{\{y\}}$. Then the T_0 -identification space of (X, T) is $(X_0, Q(X_0))$, where X_0 is the set of equivalence classes of R and $Q(X_0)$ is the decomposition topology on X_0 , which is T_0 .

This author [9] used T_0 -identification spaces to show that hyperspaces of R_0 spaces, spaces which were first defined and investigated by Davis [10], are T_1 .

DEFINITION 3.3. A space (X, T) is R_0 iff for each $0 \in T$ and $x \in 0$, $\overline{\{x\}} \subset 0$.

Since T_1 implies semi- T_1 , then the hyperspace of each R_0 space is semi- T_1 .

Semi open sets were used by Crossley and Hildebran [11] to define and investigate semi closed sets and semi closure.

DEFINITION 3.4. Let (X, T) be a space and let $A, B \subset X$. Then A is semi closed iff $X - A$ is semi open and the semi closure of B , denoted by $scl B$, is the intersection of all semi closed sets containing B .

This author [12] used semi closure to define and investigate semi- T_0 -identification spaces.

DEFINITION 3.5. Let R be the equivalence relation on a space (X, T) defined by xRy iff $scl\{x\} = scl\{y\}$. Then the semi- T_0 -identification space of (X, T) is $(X_{S0}, Q(X_{S0}))$, where X_{S0} is the set of equivalence classes of R and $Q(X_{S0})$ is the decomposition topology on X_{S0} , which is semi- T_0 .

This author [13] and [12] showed that the natural map $P: (X, T) \rightarrow (X_0, Q(X_0))$ is continuous, closed, open, onto, and $P^{-1}(P(0)) = 0$ for all $0 \in T$ and that the natural map $P_S: (X, T) \rightarrow (X_{S0}, Q(X_{S0}))$ is continuous, closed, open, onto, and $P_S^{-1}(P_S(0)) = 0$ for all $0 \in S0(X, T)$. These results are used to obtain the following result.

THEOREM 3.1. For a space $(X, T), (S(X), E(X)), (S(X_0), E(X_0))$, and $(S(X_{S0}), E(X_{S0}))$ are homeomorphic.

PROOF: Let $f: (S(X), E(X)) \rightarrow (S(X_0), E(X_0))$ and let $f_S: (S(X), E(X)) \rightarrow (S(X_{S0}), E(X_{S0}))$ defined by $f(F) = P(F)$ and $f_S(F) = P_S(F)$. Then f and f_S are homeomorphisms.

THEOREM 3.2. If (X, T) is $R_0, G \in T$, and $F \in S(X)$ such that $F \cap \overline{G} \neq \emptyset$, then $S(\overline{G}) = \overline{S(G)}$ and $F \in \overline{I(G)}$.

PROOF: Since $S(G) \subset S(\overline{G})$, which is closed, then $\overline{S(G)} \subset S(\overline{G})$. Let $A \in S(\overline{G})$. Let $\langle B_i \rangle_{i=1}^P \in \mathcal{B}$ such that $A \in \langle B_i \rangle_{i=1}^P$. Then $A \subset \overline{G}$ and

$\emptyset \neq A \cap B_i \subset \overline{G} \cap B_i$ for all $i \in N_p$, which implies $G \cap B_i \neq \emptyset$ for all $i \in N_p$.

For each $i \in N_p$ let $x_i \in G \cap B_i$. Then $\overline{\{x_i\}} \subset G \cap B_i$ for all $i \in N_p$ and

$\bigcup_{i \in N_p} \overline{\{x_i\}} \in S(G) \cap \langle B_i \rangle_{i=1}^P$. Thus $A \in \overline{S(G)}$ and $S(\overline{G}) \subset \overline{S(G)}$, which implies $S(\overline{G}) = \overline{S(G)}$.

Let $\langle U_i \rangle_{i=1}^m \in B$ such that $F \in \langle U_i \rangle_{i=1}^m$. Then $F \subset \bigcup_{i \in N_m} U_i \in T$ and $F \cap \overline{G} \neq \emptyset$, which implies $G \cap \left(\bigcup_{i \in N_m} U_i \right) \neq \emptyset$. Let $y \in G \cap \left(\bigcup_{i \in N_m} U_i \right)$ and for each $i \in N_m$ let $y_i \in B_i$. Then $\{\overline{y}\} \cup \left\{ \bigcup_{i \in N_m} \overline{\{y_i\}} \right\} \in I(G) \cap \langle U_i \rangle_{i=1}^m$. Hence, $F \in \overline{I(G)}$.

THEOREM 3.3. If (X, T) is s -regular and R_0 , then $(S(X), E(X))$ is semi- T_2 .

PROOF: Let $A, B \in S(X)$ such that $A \neq B$. Then $A - B \neq \emptyset$ or $B - A \neq \emptyset$, say $B - A \neq \emptyset$. Let $x \in B - A$. Then there exists disjoint semi open sets O and W such that $x \in O$ and $A \subset W$. Let $U, V \in T$ such that $U \subset O \subset \overline{U}$ and $V \subset W \subset \overline{V}$. Then $I(U)$ and $S(V)$ are disjoint open sets, $B \in \overline{I(U)}$, and $A \in S(\overline{V}) = \overline{S(V)}$, which implies $S(V) \cup \{A\}$ and $I(U) \cup \{B\}$ are disjoint semi open sets.

Maheshwari and Prasad [4] showed that every s -normal R_0 space is s -regular. This result can be combined with Theorem 3.3 to obtain the following corollary.

COROLLARY 3.1. If (X, T) is s -normal and R_0 , then $(S(X), E(X))$ is semi- T_2 .

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