THE CONVOLUTION-INDUCED TOPOLOGY ON $L_\infty(G)$ AND LINEARLY DEPENDENT TRANSLATES IN $L_1(G)$

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ABSTRACT. Given a locally compact Hausdorff group $G$, we consider on $L_\infty(G)$ the $\tau_c$-topology, i.e. the weak topology under all convolution operators induced by functions in $L_1(G)$. As a major result we characterize the trigonometric polynomials on a compact group as those functions in $L_1(G)$ whose left translates are contained in a finite-dimensional set. From this, we deduce that $\tau_c$ is different from the $w^\lambda$-topology on $L_\infty(G)$ whenever $G$ is infinite. As another result, we show that $\tau_c$ coincides with the norm-topology if and only if $G$ is discrete. The properties of $\tau_c$ are then studied further and we pay attention to the $\tau_c$-almost periodic elements of $L_\infty(G)$.

KEY WORDS AND PHRASES. Locally compact group, convolution operator, topology induced by convolution, linearly dependent translates, almost periodic functions.

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1. INTRODUCTION.

The reader intending to read the following paper should have some familiarity with such basic texts as Hewitt and Ross or Dunford and Schwartz.

For a locally compact Abelian group $G$, Argabright and Gil de Lamadrid [1] considered almost periodicity of measures with respect to several topologies. A special case of this general notion, namely almost periodicity with respect to the $\tau_c$-topology on $L_\infty(G)$, has been used in Crombez and Govaerts [2] in order to characterize those multipliers from $L_1(G)$ to $L_\infty(G)$ which are almost periodic in the strong
operator topology. Throughout this paper, unless explicitly stated otherwise, \( G \) will denote a locally compact Hausdorff group with left Haar measure. For such an arbitrary \( G \) the \( \tau_c \)-topology is not weaker than the \( \mathcal{W} \)-topology and not stronger than the norm topology on \( L_\infty(G) \). The question as to whether there are neighborhoods in the \( \tau_c \)-topology which are also neighborhoods in the \( \mathcal{W} \)-topology leads us to consider the apparently completely different problem of determining those functions \( f \neq 0 \) in \( L_1(G) \) such that all left translates of \( f \) are in a finite-dimensional subspace of \( L_1(G) \) (a related problem was recently investigated in Edgar and Rosenblatt [3] for Abelian groups). We prove that such functions only exist for compact \( G \), and then they are exactly the trigonometric polynomials. From this result we derive that the \( \tau_c \)-topology is always different from the \( \mathcal{W} \)-topology whenever \( G \) is infinite. However, a further investigation shows that for compact \( G \) these two topologies coincide on every norm-bounded subset of \( L_\infty(G) \), and so we may conclude that for compact \( G \)

\( L_1(G) \) is the dual of \( (L_\infty(G), \tau_c) \). Among the other results we mention that except for discrete \( G \) the \( \tau_c \)-topology is always different from the norm-topology (section 3), and that for fixed \( g \) in \( L_\infty(G) \) the map \( s \mapsto g \) from \( G \) to \( (L_\infty(G), \tau_c) \) is continuous (section 4). In section 5 we give some further results about \( \tau_c \)-almost periodic functions.

For complex-valued functions \( f \) and \( g \) on \( G \) and \( a \in G \), we define the left translate \( a^f \) and the convolution \( f*g \) by means of \( a^f(x) = f(ax) \) and \( (f*g)(x) = \int_G f(xy)g(y^{-1})dy \) (we warn the reader that in some of the references, e.g. [4] and [5], different conventions are used). Each function \( f \) in \( L_1(G) \) induces by convolution an operator \( T_f \) on \( L_\infty(G) \); the weak topology on \( L_\infty(G) \) under all convolution operators \( T_f : L_\infty(G) \to (L_\infty(G), || ||_\infty) \) is denoted by \( \tau_c \). By \( \mathcal{W} \) and \( || ||_\infty \) we denote the \( (L_\infty(G), L_1(G)) \), i.e., weak \( * \) topology, and the essential supremum norm topology respectively, on \( L_\infty(G) \). All other nonexplained notation is taken from Hewitt and Ross [6].

2. Functions in \( L_1(G) \) with finite-dimensional span of translates.

From the definitions we immediately derive \( \mathcal{W} \leq \tau_c \leq || ||_\infty \). Investigation of the possibility that some \( \tau_c \)-neighborhood is also a \( \mathcal{W} \)-neighborhood leads to a special class of functions in \( L_1(G) \), as Proposition 1 shows. For convenience we take as a subbase at 0 for \( \mathcal{W} \) the sets
\{h \in L^\infty(G) : \left| \int_G f(x)h(x^{-1})dx \right| < \varepsilon\}, \text{ where } f \in L^1(G) \text{ and } \varepsilon > 0; \text{ we write } \langle f, h \rangle \text{ for } \int_G f(x)h(x^{-1})dx.\\

PROPOSITION 1. For \(0 \neq f \in L^1(G)\) the following are equivalent:

(i) There exists an \(\varepsilon > 0\) such that the \(\tau_c\)-neighborhood determined by \(f\) and \(\varepsilon\) is a \(w^\ast\)-neighborhood.

(ii) The set of left translates of \(f\) is part of a finite-dimensional subspace of \(L^1(G)\).

(iii) There exist \(a_1, \ldots, a_n \in G\) such that, for each \(a \in G\), scalars \(c_1, \ldots, c_n\) may be found such that \(f = \sum_{i=1}^n c_i a_i f\).

(iv) Given \(\varepsilon > 0\), there exists \(a_1, \ldots, a_n \in G\) and \(\delta > 0\) such that, for \(g \in L^\infty(G)\), the inequality \(\left| \langle a, f, g \rangle \right| < \delta\) for all \(i = 1, \ldots, n\) implies \(\|f \cdot g\|_\infty < \varepsilon\).

PROOF (i) \(\Rightarrow\) (ii). Suppose that the set \(\{g \in L^\infty(G) : \|f \cdot g\|_\infty < \varepsilon\}\) is a \(w^\ast\)-neighborhood of zero. Then we may find functions \(f_i (i = 1, \ldots, r)\) in \(L^1(G)\) and \(\delta > 0\) such that, whenever \(g \in L^\infty(G)\) and \(\left| \int_G f_i(x)g(x^{-1})dx \right| < \delta\) for all \(i\), then \(\|f \cdot g\|_\infty < \varepsilon\). Each \(f_i\) determines a linear functional on \(L^\infty(G)\); call \(N\) the intersection of their kernels. Since for any scalar \(c\), \(cg \in N\) whenever \(g \in N\), there exists that \(|c| \|f \cdot g\|_\infty < \varepsilon\) for \(g \in N\) and for any scalar \(c\); hence \(f \cdot g = 0\) for \(g\) in \(N\), or \(\int_G f(y)g(y^{-1})dy = 0\) for any \(a \in G\) and \(g \in N\). This means that, for given \(a \in G\), the linear functional determined by \(f\) may be written as a linear combination of the ones determined by the \(f_i (i = 1, \ldots, r)\). So, given \(a \in G\), there exist scalars \(a_1, \ldots, a_r\) such that \(a f = \sum_{i=1}^r a_i f_i\).

(ii) \(\Rightarrow\) (iii). Obvious. We may choose \(a_1, \ldots, a_n \in G\) such that the set \(\{a f\}_{a \in G}^{n, i=1}\) is also a linearly independent set.

(iii) \(\Rightarrow\) (iv). We first remark that the assumption of (iii) implies that \(G\) is necessarily compact. Indeed, whenever (iii) is true the set \(\{a f : a \in G\}\) of left translates of \(f\) is a norm-bounded subset of a finite-dimensional subspace of \(L^1(G)\), and so this set is relatively compact with respect to the norm-topology of \(L^1(G)\). However, it was shown in Crombez and Govaerts [4] that for non-compact \(G\) only \(f = 0\) has this property.

From this it also follows that there exists \(B > 0\) such that for all \(a \in G\), \(\sum_{i=1}^n |c_i| < B\)
for the scalars figuring in (iii). Indeed, the function $f_a$ from $G$ to $L^1(G)$ is continuous, and its range is part of a finite-dimensional subspace $M$ of $L^1(G)$; assuming, as we may, that $\{a_1 f\}_{i=1}^n$ is linearly independent, the function $f = \sum_{i=1}^n c_i a_i f_a(c_1,\ldots,c_n)$ from $M$ to the $n$-dimensional complex space $\mathbb{C}^n$ is (well-defined and) linear, and hence continuous; so the composition of these two functions is continuous on the compact group $G$, from which the result follows.

Suppose then that (iii) is true, and let $\varepsilon > 0$ be given. Choose $\delta > 0$ such that $B_\delta < \varepsilon$, with $B$ as mentioned above. If $a_1,\ldots,a_n$ are as in (iii) and $|<a_1 f,g>| < \delta$ for all $i=1,\ldots,n$, then for $a,\bar{a} \in G$ we have

$$|(f*g)(a)| = \left|\sum_{i=1}^n c_i a_i f(y)g(y^{-1})dy\right| < \varepsilon \sum_{i=1}^n |c_i| |<a_1 f,g>| < \varepsilon.$$ 

(iv)$\Rightarrow$(i). Obvious.

Statement (iii) in Proposition 1 leads to the following problem: determine those $f \in L^1(G)$ for which all left translates are contained in a finite-dimensional subset of $L^1(G)$. As remarked in the proof of the proposition such nonzero functions can exist only for compact $G$. To solve this problem, we use the theory of representations of compact groups as explained in Hewitt and Ross [6]. It is readily verified that the set of functions with the mentioned property is a linear subspace $V$ of $L^1(G)$ containing all trigonometric polynomials. Proposition 2 shows that there are no other functions in $V$. (For related results in the abelian case, we refer to Schwartz [7], and to the recent paper of Laird [8] and the references mentioned there.)

PROPOSITION 2. Let $0 \neq f \in L^1(G)$ with $G$ compact. The set $\{a_1 f : a \in G\}$ of left translates of $f$ is contained in a finite-dimensional space iff $f$ is a trigonometric polynomial on $G$.

PROOF. We first remark that $f$ is a trigonometric polynomial iff the Fourier transform $\hat{f}$ of $f$ is such that $\hat{f}(\sigma) = 0$ except for a finite number of elements $\sigma$ in the dual object $\hat{G}$ of $G$ (see Hewitt and Ross [6], 28.39).

Let then $f \in L^1(G)$ be such that statement (iii) of Proposition 1 is true, i.e., $f_a = \sum_{i=1}^n c_i(a) f$ (for fixed $n$) and $\sum_{i=1}^n |c_i(a)| < B$ (this was shown in the proof of (iii)$\Rightarrow$(iv) above). Taking the Fourier transform we obtain
[\overline{U}_a^{(a)} - \sum_{i=1}^{n} c_i(a) \overline{U}_a^{(a)}] \hat{f}(\sigma) = 0. Let D be the set of those \sigma \in \mathbb{Z} for which \hat{f}(\sigma) is different from zero. Then for each \sigma \in D there is a subspace \mathcal{M}_\sigma \neq \{0\} in the representation space \mathcal{H}_\sigma of \overline{U}_G^{(a)} such that \overline{U}_a^{(a)} = \sum_{i=1}^{n} c_i(a) \overline{U}_a^{(a)} = 0 on \mathcal{M}_\sigma, \forall \sigma \in G. We choose an element \xi^{(a)} in \mathcal{M}_\sigma with \|\xi^{(a)}\| = 1. Since \overline{U}_G^{(a)} is irreducible, the non-zero vector \xi^{(a)} is a cyclic vector for \overline{U}_G^{(a)} which means that the set of all finite linear combinations of elements from \{\overline{U}_a^{(a)}\xi^{(a)} : a \in G\} is all of \mathcal{H}_\sigma; but the set \{\overline{U}_a^{(a)}\xi^{(a)} : a \in G\} is spanned by the finitely many vectors \overline{U}_a^{(a)}\xi^{(a)} = \sigma_1, \ldots, \overline{U}_a^{(a)}\xi^{(a)}; hence, if d_\sigma denotes the dimension of \mathcal{H}_\sigma we always have d_\sigma \leq n, for each \sigma in D.

With the choice of \xi^{(a)} we have \|\langle \overline{U}_a^{(a)}\xi^{(a)}, \xi^{(a)} \rangle\| < 1 for all \sigma \in D and all i \in \{1, \ldots, n\}, where now \langle, \rangle denotes the inner product on \mathcal{H}_\sigma. If D is infinite, we obtain an infinite family \{\langle \overline{U}_a^{(a)}\xi^{(a)}, \xi^{(a)} \rangle, \ldots, \langle \overline{U}_a^{(a)}\xi^{(a)}, \xi^{(a)} \rangle\}_{\sigma \in D} in a compact set in the n-dimensional complex space, and so it has a cluster point; this means that, given 0 < \frac{1}{d_n}, there exist different \sigma_1 and \sigma_2 in D such that

\|\langle \overline{U}_a^{(a)}\xi^{(a)}, \xi^{(a)} \rangle - \langle \overline{U}_a^{(a)}\xi^{(a)}, \xi^{(a)} \rangle\| \leq \varepsilon for all i.

For each a in G we then have

\|\langle \overline{U}_a^{(a)}\xi^{(a)}, \xi^{(a)} \rangle - \langle \overline{U}_a^{(a)}\xi^{(a)}, \xi^{(a)} \rangle\| = \left| \sum_{i=1}^{n} c_i(a) \langle \overline{U}_a^{(a)}\xi^{(a)}, \xi^{(a)} \rangle - \langle \overline{U}_a^{(a)}\xi^{(a)}, \xi^{(a)} \rangle \right| \leq \varepsilon

Assuming that the Haar measure of the compact group G is normalised, it follows that

\left| \int_{\mathbb{G}} \langle \overline{U}_a^{(a)}\xi^{(a)}, \xi^{(a)} \rangle - \langle \overline{U}_a^{(a)}\xi^{(a)}, \xi^{(a)} \rangle \overline{U}_a^{(a)}\xi^{(a)}, \xi^{(a)} \rangle da \right| \leq \varepsilon,

while the first member has the value \frac{1}{d_{\sigma_1}}. Since d_\sigma \leq n (fixed), we arrive at a contradiction by our choice of \varepsilon. \[\square\]

3. CONNECTION OF \(\tau_c\) WITH OTHER TOPOLOGIES ON \(L^\infty(G)\).

From Proposition 1 we immediately conclude that for non-compact G the \(\tau_c\)-topology is always strictly weaker than the \(\tau_c\)-topology. But taking Proposition 2 into account, we infer that also for infinite compact G these two topologies are different. Indeed, it suffices to remark that there always exists a function f
in $L_1(G)$ which is not a trigonometric polynomial (e.g., choose in $\sum$ a countable infinite set $\{\sigma_n\}_{n=1}^\infty$ of different elements; let $\chi_{\sigma_n}$ be the corresponding character, $n \in \omega$ and put $f(x) = \sum_{n=1}^{\infty} \frac{\chi_{\sigma_n}(x)}{n^2d_{\sigma_n}^2}$ for $x \in G$; then $f \in L_1(G)$, and $f(\sigma_n) = \frac{1}{n^2d_{\sigma_n}^2} \int_{\sigma_n}^G$, where $I_{H_{\sigma_n}}$ is the identity operator on $H_{\sigma_n}$.

Although $\tau_c$ and $\omega^\infty$ are different for infinite compact $G$, they induce the same topology on every norm-bounded subset of $L_\infty(G)$, as the following proposition shows.

**Proposition 3.** If $G$ is compact, and $B$ is a norm-bounded subset of $L_\infty(G)$, then $\tau_c$ and $\omega^\infty$ coincide on $B$.

**Proof.** It is sufficient to prove that for any $\tau_c$-neighborhood $V$ of 0 there exists a $\omega^\infty$-neighborhood $W$ of 0 such that $W \cap B = V$. Suppose that $\|h\|_\infty < M, \forall h \in B$, and let $V = \{h \in L_\infty(G): |\int f_i h|_\infty < \varepsilon \text{ for } i = 1, \ldots, n\}$ with given $f_i \in L_1(G)$ and $\varepsilon > 0$. From compactness of $G$ and continuity of $a^f f$ from $G$ to $(L_1(G), \|\|_1)$ it follows that each $f_i$ is almost periodic in $(L_1(G), \|\|_1)$; this means that there exists elements $a_1, \ldots, a_m$ in $G$ such that, for each $a$ in $G$ and each $i \in \{1, \ldots, n\}$ a point $a_j$ may be found ($1 \leq j \leq m$) such that $|a^f_i - a_j^f_i|_1 < \varepsilon$. With this choice of $a_j$ and for $g \in L_\infty(G)$ we have

$$|\int (f_i h)(a) - (f_i h)(a_j)| \leq |\int a^f_i - a_j^f_i|_1 \|g\|_\infty, \text{ or for } g \in B, |\int a^f_i h(a)|_1 \leq |\int a_j^f_i h(a)|_2 < \frac{\varepsilon}{2}.$$ 

Put $W = \{h \in L_\infty(G): |\int a_j^f_i h|_2 < \varepsilon \text{ for all } i, j\}$. Then $W$ is a $\omega^\infty$-neighborhood of 0, and for $h$ in $W \cap B$ we obtain $\|f_i h\|_\infty < \varepsilon$.

**Corollary 1.** For compact $G$, any $\omega^\infty$-convergent sequence is $\tau_c$-convergent. Indeed, the set consisting of the elements in the sequence together with its limit is $\omega^\infty$-compact, and hence also norm bounded.

**Corollary 2.** For compact $G$, $L_1(G)$ is the dual of $(L_\infty(G), \tau_c)$.

**Proof.** For a compact group $G$ there is a connection between the $\tau_c$-topology and the so-called bounded weak$^\infty$-topology $bw^\infty$ (see Holmes [9], p. 150; this topology is called the bounded $X$-topology in Dunford and Schwartz [10], p. 427); indeed, we have $\tau_c \leq bw^\infty$. The result then follows from the fact that $L_1(G)$ is the dual of $(L_\infty(G), bw^\infty)$.

The following proposition characterizes those groups for which $\tau_c$ and $\|\|_\infty$
PROPOSITION 4. $\tau_c$ coincides with $|| \cdot ||_\infty$ iff $G$ is discrete.

PROOF. For discrete $G$ we have that the $\tau_c$-topology and the $|| \cdot ||_\infty$-topology are equal. For if $e$ is the identity of $G$, then $\delta_e$ is a convolution identity for $L_1(G)$, and the convolution operator $T_{\delta_e}$ induced by $\delta_e$ is the identity map on $L_\infty(G)$.

Let then $G$ be non-discrete. Given $\varepsilon > 0$ and $f_1, \ldots, f_n$ in $L_1(G)$, choose a compact subset $K$ in $G$ such that $\int |f_i(x)|dx < \frac{\varepsilon}{2}$, for each $i=1,\ldots,n$. Let $\eta > 0$ be such that $G/K$ is a compact symmetric neighborhood of the identity $e$ of $G$ with $0 < \mu(K) < \eta$. Further, let $U$ be a compact symmetric neighborhood of the identity $e$ of $G$ with $0 < \mu(U) < \eta$. Let $g$ be the function defined by $g(x) = 1$ for $x \in U$, and $g(x) = 0$ on $G \setminus U$. For $1 \leq i \leq n$ and $x \in G$ we obtain

$$|f_i(x)| < \frac{\varepsilon}{2}$$

for all $i=1,\ldots,n$ and all measurable $A \subseteq G$ with $\mu(A) < \eta$, where $\mu$ denotes left Haar measure. Further, let $U$ be a compact symmetric neighborhood of the identity $e$ of $G$ with $0 < \mu(U) < \eta$. Let $g$ be the function defined by $g(x) = 1$ for $x \in U$, and $g(x) = 0$ on $G \setminus U$. For $1 \leq i \leq n$ and $x \in G$ we obtain

$$|f_i(x)| < \frac{\varepsilon}{2}$$

Both terms on the right-hand side are dominated by $\frac{\varepsilon}{2}$, since $g(y^{-1}x)$ is zero except when $y \in xU^{-1}$, and since $\mu(xU^{-1}) < \eta$. Hence $||f_i g||_\infty < \varepsilon$, although $||g||_\infty = 1$. This shows that no $\tau_c$-neighborhood of $0$ lies wholly in any $|| \cdot ||_\infty$-ball of radius less than $1$. Hence $\tau_c$ is coarser than $|| \cdot ||_\infty$.

4. FURTHER PROPERTIES OF THE $\tau_c$-TOPOLOGY.

The proofs of Propositions 5 and 8 that follow were kindly suggested to us by Robert B. Burckel. Both results also appear in Crombez and Govaerts[2].

PROPOSITION 5. Any norm-closed ball in $L_\infty(G)$ is $\tau_c$-complete.

PROOF. Let $(g_\alpha)$ be a $\tau_c$-Cauchy net in a ball in $L_\infty(G)$. Let $g$ be a $w^\infty$-cluster point of this net, such that a subnet $(g_{\alpha'})$ $w^\infty$-converges to $g$. Then $\{(fg_{\alpha'})\}(x)$ converges to $(fg)(x)$ for all $x \in G$ and all $f \in L_1(G)$. Given $\varepsilon > 0$ and $f \in L_1(G)$, there exists $\alpha_\varepsilon$ such that $||f g_{\alpha_\varepsilon} - f g_\alpha||_\infty < \varepsilon$ for all $\alpha, \alpha' \geq \alpha_\varepsilon$. Since all these functions are continuous and $|| \cdot ||_\infty$ here is genuine supremum, we derive

$$|(fg_\alpha)(x) - (fg_\alpha')(x)| < \varepsilon$$

for all $\alpha, \alpha' \geq \alpha_\varepsilon$, for all $x \in G$. In this last inequality
we take $\alpha' = \beta$ and let $\beta$ recede to infinity; then this leads to
\[
|f^{(\alpha')}(x) - f^{(\alpha)}(x)| < \varepsilon \text{ for all } \alpha' > \alpha \text{ and all } x \in G, \text{ i.e.,}
\]
\[
|f^{(\alpha')}(x) - f^{(\alpha)}(x)| \leq \varepsilon \text{ for all } \alpha' > \alpha =: \alpha.
\]

In particular, we derive from Proposition 5 that a set in $L(G)$ is $\mathcal{T}_c$-relatively compact iff it is $\mathcal{T}_c$-totally bounded. We also have that the closed absolutely convex hull of a $\mathcal{T}_c$-compact set is again $\mathcal{T}_c$-compact. Denoting by $\text{cl}_{\mathcal{T}_c}$ the closure in the $\mathcal{T}_c$-topology, we have

**Proposition 6.** If $g \in L(G)$, then $g \in \text{cl}_{\mathcal{T}_c}(L_1 \ast g)$.

**Proof.** Given $\varepsilon > 0$ and $n$ functions $k_i$ in $L_1(G)$ determining a $\mathcal{T}_c$-neighborhood $V$ of $g$ in $L(G)$, and denoting by $\{e_\lambda\}_{\lambda \in \Lambda}$ an approximate identity in $L_1(G)$, we see that $\|k_i \ast (e_\lambda \ast g) - k_i \ast g\|_\infty$ may be made arbitrarily small. Hence $V$ contains elements of the form $e_\lambda \ast g$.

**Corollary 3.** Let $S$ be a $\mathcal{T}_c$-closed $L_1$-submodule of $L(G)$. Then $S = \text{cl}_{\mathcal{T}_c}(L_1 \ast S)$.

**Corollary 4.** Let $S$ be a $\mathcal{T}_c$-closed $L_1$-submodule of $L(G)$. Then $S$ is left translation invariant.

**Proof.** Given $g \in S$ and $a \in G$ we show that any $\mathcal{T}_c$-neighborhood of $a \ast g$ contains a function in $L_1 \ast S$, from which the result will follow. Denote by $\Delta$ the modular function of $G$. Let $V$ be the $\mathcal{T}_c$-neighborhood of $a \ast g$ determined by $f_1, \ldots, f_n$ in $L_1(G)$ and $\varepsilon > 0$. There always exist $k \in L_1(G)$ and $h \in S$ such that
\[
\|f_1 \ast a \ast k - f_1 \ast k \ast h\|_\infty < \frac{\varepsilon}{\Delta(a)}.
\]
Then $\|f_1 \ast g - f_1 \ast (k \ast h)\|_\infty < \varepsilon$. Hence $V$ contains the function $a \ast k \ast h \in L_1 \ast S$.

Since $w < \mathcal{T}_c$, Proposition 6 and its corollaries are stronger than the corresponding results in Crombez and Govaerts [5].

Given $g \in L(G)$, the map $s \mapsto s \ast g$ from $G$ to $(L(G), \| \cdot \|_\infty)$ is continuous iff $g$ is locally a.e. equal to a function in $C_{ru}(G)$. (Here, as in [11], $C_{ru}(G)$ is the set of all right uniformly continuous, bounded, complex-valued functions on $G$). However, using the $\mathcal{T}_c$-topology on $L(G)$ we obtain continuity for any $g \in L(G)$.
PROPOSITION 7. Let \( g \) be a function in \( L^\infty(G) \). Then the maps \( s \mapsto g_s \) and \( s^+ \mapsto g_s \) from \( G \) to \( (L^\infty(G), \tau_c) \) are continuous.

PROOF. That the map \( s \mapsto g_s \) is continuous is trivial, since for any \( f \) in \( L^1(G) \), \( f \in C_r^\ast(G) \) and \( f g_s = (f g)_s \). To prove that \( s^+ \mapsto g_s \) is continuous, consider the composition of the maps \( G \to L^1(G) \times C_r^\ast(G) \) given by \( s^+((f_s, \Delta(s)) + \Delta(s)f g_s = f g_s \).

Each map is continuous, and so the result follows.

5. SOME MORE RESULTS ON \( \tau_c \)-ALMOST PERIODIC FUNCTIONS.

In this final section we always suppose \( G \) to be Abelian. The notion of \( \tau_c \)-almost periodic (\( \tau_c \)-AP) function in \( L^\infty(G) \) was introduced in [2] in order to characterize those multipliers which are strongly almost periodic.

PROPOSITION 8. A function \( g \) in \( L^\infty(G) \) is \( \tau_c \)-AP iff \( f g \) is \( || \cdot ||_\infty \)-almost periodic for each \( f \) in \( L^1(G) \).

PROOF. We first notice that \( f g_a = (f g)_a \) for any \( a \) in \( G \); so if we set \( 0_g = \{ g : a \in G \} \), then \( f g 0 = 0_g f g \).

If \( 0_g \) is relatively \( \tau_c \)-compact, then its continuous image \( f g 0 = 0_g f g \) in \( (C_r^\ast(G), || \cdot ||_\infty) \) is relatively compact, so \( f g \) is norm almost periodic. Conversely, by definition of \( \tau_c \) the map

\[
\tau_c((f g)_f) \in \prod_{f \in L^1(G)} f g L^\infty(G)
\]

is a homeomorphism from \( \tau_c \) into the product of the norm topologies on the right. Evidently the image of \( 0_g \) lies in the subspace \( \prod_{f \in L^1(G)} 0_g f g \). If each \( f g \) is norm almost periodic, then this last product is relatively compact, and so \( 0_g \) is relatively \( \tau_c \)-compact.

Denoting by \( \text{AP} \) the \( || \cdot ||_\infty \)-almost periodic functions in \( L^\infty(G) \), we obtained in [2] that \( \tau_c \)-AP = \( \text{AP} \) for \( G \) discrete, and \( \tau_c \)-AP = \( L^\infty(G) \) for \( G \) compact (both results are of course clear now by Proposition 4 and Proposition 3, respectively). We always have that \( \text{AP} \subseteq \tau_c \)-AP. From Proposition 8 we derive: \( L^1(G) \tau_c \)-AP \( \subset \text{AP} \). Since \( L^1(G) \tau_c \)-AP = \( \text{AP} \) (see Crombez and Govaerts [4]), we also get \( L^1(G) \tau_c \)-AP = \( \text{AP} \). Hence we obtain from Proposition 8 that \( \tau_c \)-AP is the largest linear subspace \( S \) of \( L^\infty(G) \).
such that $L_1(G) \# S = AP$. The set $\tau_c^{-1} - AP$ is an $L_1$-submodule of $L_\infty(G)$ which is obviously $\tau_c$-closed. From Corollary 3 we may conclude that $\tau_c^{-1} - AP = cl_{\tau_c} AP$. In particular, for compact $G$ we have that $L_\infty(G) = cl_{\tau_c} C(G)$, where $C(G)$ denotes the set of continuous functions on $G$.

**PROPOSITION 9.** $G$ is compact iff $\tau_c^{-1} - AP = L_\infty(G)$.

**PROOF.** Suppose that $\tau_c^{-1} - AP = L_\infty(G)$. Then $AP = L_1(G) \# \tau_c^{-1} - AP = L_1(G) \# L_\infty(G) = C_{ru}(G)$, the last equality coming from Hewitt and Ross [6], 32.45(b). Pick $0 \neq f \in C_{ru}(G)$ with compact support $K$. If $G$ is not compact there exist infinitely many disjoint translates $a.K$ of $K$. Clearly the subset $\{ \frac{-1}{a} f \}_{j=1}^\infty$ of the left orbit of $f$ is not totally bounded.

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