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ON THE PARTITION PROPERTY OF MEASURES ON $P_\kappa\lambda$

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ABSTRACT. The partition property for measures on $P_\kappa\lambda$ was formulated by analogy with a property which Rowbottom [1] proved was possessed by every normal measure on a measurable cardinal. This property has been studied in [2], [3], and [4]. This note summarizes [5] and [6], which contain results relating the partition property with the extendibility of the measure and with an auxiliary combinatorial property introduced by Menas in [4]. Detailed proofs will appear in [5] and [6].

KEY WORDS AND PHRASES. Supercompact cardinals, measures with the partition property, extendible measures.

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1. DEFINITIONS AND SUMMARY OF KNOWN RESULTS.

Where κ,λ are infinite cardinals with $\kappa \leq \lambda$, $P_\kappa\lambda$ denotes the set $\{p \subseteq \lambda \mid |p| < \kappa\}$. A <u>measure</u>, μ, on $P_\kappa\lambda$ is a $\{0,1\}$-valued function defined on subsets of $P_\kappa\lambda$ satisfying

$\forall\alpha < \lambda$, $\mu(\{\alpha\}) = 0$ (non-triviality); (1)

$\mu(\emptyset) = 0$ and $\mu(P_\kappa\lambda) = 1$; (2)

if $\gamma < \kappa$ and $\{X_\alpha \mid \alpha < \gamma\}$ is pairwise disjoint, then $\mu(\bigcup_{\alpha<\gamma} X_\alpha) = \sum_{\alpha<\gamma} \mu(X_\alpha)$ (κ-additivity). (3)

A measure μ on $P_\kappa\lambda$ is called <u>normal</u> if, in addition, it satisfies

$\forall f: P_\kappa\lambda \to \lambda$ such that $\mu(\{p \in P_\kappa\lambda \mid f(p) \in p\}) = 1$, $\exists X \subseteq P_\kappa\lambda$ with $\mu(X) = 1$ and $\exists\alpha < \lambda$ such that $\forall p \in X$, $f(p) = \alpha$.

κ is called λ-<u>supercompact</u> if there exists a normal measure on $P_\kappa\lambda$.

κ is <u>supercompact</u> if it is λ-supercompact $\forall\lambda \geq \kappa$.

This concept was introduced in [7] because of its analogy with the notion of a measurable cardinal. (A cardinal κ is measurable if there exists a $\mu : P(\kappa) \to \{0,1\}$ satisfying (1) - (3) above.) Many questions about measures on $P_\kappa\lambda$ are motivated by attempts to generalize results that are known for measurable cardinals.

A measure μ on the measurable cardinal κ is called <u>normal</u> if $\forall f: \kappa \to \kappa$ such that $\mu(\{\gamma < \kappa | f(\gamma) < \gamma\}) = 1$, $\exists X \subseteq \kappa$ with $\mu(X) = 1$ and $\exists\alpha < \kappa$ such that $\forall\gamma \in X$, $f(\gamma) = \alpha$.

The theorem that motivates the partition property is the following one of Rowbottom [1]: if μ is a normal measure on a measurable cardinal κ and if $F : \{\{\alpha,\beta\} | \alpha < \beta < \kappa\} \to \{0,1\}$, then $\exists H \subseteq \kappa$ with $\mu(H) = 1$ and $\exists i \in \{0,1\}$ such that $F(\{\alpha,\beta\}) = i$ whenever $\alpha < \beta$ and $\alpha,\beta \in H$.

Where $X \subseteq P_\kappa\lambda$, let $[X]^2 = \{\{p,q\} | p,q \in X \;\&\; p \neq q\}$. A subset $X \subseteq P_\kappa\lambda$ is called <u>homogeneous for F</u> , where $F : [P_\kappa\lambda]^2 \to \{0,1\}$, if F is constant on $\{\{p,q\} \in [X]^2 | p \subset q \text{ or } q \subset p\}$. A normal measure μ on $P_\kappa\lambda$ is said to <u>have the partition property</u> (which we abbreviate by Part (μ)) if $\forall F: [P_\kappa\lambda]^2 \to \{0,1\}$, $\exists X \subseteq P_\kappa\lambda$ with $\mu(X) = 1$ which is homogeneous for F .

By analogy with Rowbottom's theorem, it was natural to conjecture that every normal measure on $P_\kappa\lambda$ would have the partition property. The following list of theorems summarizes the results that are known.

THEOREM A (Menas [4]). If κ is $2^{\lambda^{<\kappa}}$-supercompact, then there exist $2^{2^{\lambda^{<\kappa}}}$-many normal measures on $P_\kappa\lambda$ with the partition property.

THEOREM B (Solovay; published in [4]). If κ is supercompact and $\lambda > \kappa$ is β-supercompact for some $\beta \geq \lambda$ (in particular, if λ is measurable), then there is a normal measure on $P_\kappa\beta$ without the partition property.

THEOREM C (Solovay, assuming GCH: Menas, without GCH [4]). If κ is supercompact, then for certain small cardinals (e.g. $\lambda = \kappa^+$ or $\lambda = 2^\kappa$) every normal measure on $P_\kappa\lambda$ has the partition property.

THEOREM D (Kunen [3], details are in [6]). If κ is supercompact, then the least $\alpha > \kappa$ such that $P_\kappa\lambda$ bears a normal measure without the partition property

is Π^2_1-indescribable and inaccessible.

2. MENAS' PROPERTY χ .

In his proof of Theorem A, Menas introduced an auxiliary combinatorial property. A measure μ on $P_\kappa\lambda$ is said to have property χ (which we will abbreviate as $\chi(\mu)$) if there exists an $f: \kappa \to \kappa$ such that

$$\mu(\{p \in P_\kappa\lambda \mid f(|p \cap \kappa|) = |p| \ \& \ \forall\alpha < |p \cap \kappa| \ , \ f(\alpha) < |p \cap \kappa|\}) = 1 \ .$$

His proof of Theorem A can be broken into three lemmas.

LEMMA A_1 . Part(μ) iff $\exists X \subseteq P_\kappa\lambda$ with $\mu(X) = 1$ such that $\forall p,q \in X$ $(p \subsetneq q \to |p| < |q \cap \kappa|)$.

LEMMA A_2 . If κ is $2^{\lambda^{<\kappa}}$-supercompact, then there exist $2^{2^{\lambda^{<\kappa}}}$-many normal measures on $P_\kappa\lambda$ with property χ .

LEMMA A_3 . For any normal measure, μ , on $P_\kappa\lambda$, $\chi(\mu) \to$ Part(μ) .

His proof of Lemma A_3 uses the characterization of Part(μ) given in Lemma A_1 .

3. RESTRICTION MEASURES, EXTENDIBLE MEASURES, AND ELEMENTARY EMBEDDINGS.

Suppose $\kappa \leq \alpha \leq \beta$ and $g: P_\kappa\beta \to P_\kappa\alpha$, If μ is a measure on $P_\kappa\beta$, then $g_*(\mu)$ is the $\{0,1\}$-valued function defined on subsets of $P_\kappa\alpha$ by

$$g_*(\mu)(X) = 1 \ \text{ iff } \ \mu(\{p \in P_\kappa\beta \mid g(p) \in X\}) = 1 \ .$$

For the function $g(p) = p \cap \alpha$, it can be proved that $g_*(\mu)$ is a measure on $P_\kappa\alpha$; it is called the restriction of μ to $P_\kappa\alpha$ and is denoted by $\mu|\alpha$. In this same situation, if ν is a measure on $P_\kappa\alpha$, we say that ν is β-extendible if there exists a measure μ on $P_\kappa\beta$ such that $\nu = \mu|\alpha$.

For a normal measure, μ , on $P_\kappa\lambda$, let M_μ denote the transitive collapse of the ultrapower of the universe, $V^{P_\kappa\lambda}/\mu$, constructed from μ and let $j_\mu: V \to M_\mu$ be the corresponding elementary embedding. (See [7] for details).

THEOREM E ([7]). Let $\kappa \leq \alpha \leq \beta$, let μ be a normal measure on $P_\kappa\beta$ and let $\nu = \mu|\alpha$. Then there exists an elementary embedding $k : M_\nu \to M_\mu$ such that $k \circ j_\nu = j_\mu$.

4. THEOREMS.

THEOREM 1. ([5]). For any normal measure μ on $P_\kappa\lambda$, $\chi(\mu)$ iff there exists an $f : \kappa \to \kappa$ such that $j_\mu(f)(\kappa) = \lambda$.

This alternate characterization of property χ in terms of the behaviour of the associated elementary embedding can be used to provide easier proofs of Theorems A and C. It is also crucial to the proof of Theorem 2 which shows that Menas' property χ is not equivalent to the partition property.

THEOREM 2. ([6]). Let $\kappa < \aleph_{(2^{\kappa})^+} \leq \lambda <$ the least inaccessible cardinal greater than κ, let κ be λ-supercompact, and let μ be any normal measure on $P_\kappa\lambda$; then there exists a cardinal $\alpha \in [\kappa,\lambda)$ such that $\mathrm{Part}(\mu|\alpha)$ and $\neg\chi(\mu|\alpha)$.

The proof uses Theorem 1 and Theorem E to get $\neg\chi(\mu|\alpha)$ and Theorem D to get $\mathrm{Part}(\mu|\alpha)$.

THEOREM 3. [5]. Suppose $\phi(\alpha,\xi)$ is a formula in L_{ZF} satisfying

(i) $\phi(\alpha,\xi_1)\ \&\ \phi(\alpha,\xi_2) \rightarrow \xi_1 = \xi_2$,

(ii) $\forall\alpha < \kappa\ ,\ \exists\xi < \kappa\ \ \phi(\alpha,\xi)$;

let $\kappa \leq \lambda$ and assume that for some $\beta > \lambda$, there exists a normal measure μ on $P_\kappa\beta$ such that $M_\mu \models \phi(\kappa,\lambda)$; then $\chi(\mu|\lambda)$.

In particular, if (i) and (ii) hold for $\phi(\alpha,\xi)$ and ν is a normal measure on $P_\kappa\lambda$ which is β-extendible to a measure μ on $P_\kappa\beta$ for which $M_\mu \models \phi(\kappa,\lambda)$, then $\chi(\nu)$ (and hence $\mathrm{Part}(\nu)$ by Lemma A_3).

The proof uses Theorem 1 in conjunction with Theorem E.

Theorem 3 can be used to show that certain measures are not extendible. The existence of non-extendible measures on a supercompact cardinal is known. In fact, if κ is supercompact and if μ is minimal in the Mitchell ordering of normal measures on $P_\kappa\lambda$ (see [7] for this definition), it can be shown that μ is not $2^{\lambda^{<\kappa}}$-extendible. Here, we show that Solovay's "glue-together" measures are not extendible.

Where κ is λ-supercompact and λ is measurable, the so called "glue-together" measures are defined as follows: fix a normal measure τ on λ and a normal measure μ on $P_\kappa\lambda$, and for $X \subseteq P_\kappa\lambda$, define $\nu(X) = \tau(\{\alpha < \lambda \,|\, \mu|\alpha(X \cap P_\kappa\alpha) = 1\})$; this is a special case of the construction used by Solovay in his proof of Theorem B.

THEOREM 4. [5]. If κ is λ-supercompact where λ is the least measurable cardinal greater than κ, then the "glue-together" measures on $P_\kappa\lambda$ are not 2^λ-

extendible.

The proof uses Theorem 3 in conjunction with Theorem B and Lemma A_3 .

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