ON THE PARTITION PROPERTY OF MEASURES ON $P^\lambda_\kappa$

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ABSTRACT. The partition property for measures on $P^\lambda_\kappa$ was formulated by analogy with a property which Rowbottom [1] proved was possessed by every normal measure on a measurable cardinal. This property has been studied in [2], [3], and [4]. This note summarizes [5] and [6], which contain results relating the partition property with the extendibility of the measure and with an auxiliary combinatorial property introduced by Menas in [4]. Detailed proofs will appear in [5] and [6].

KEY WORDS AND PHRASES. Supercompact cardinals, measures with the partition property, extendible measures.

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I. DEFINITIONS AND SUMMARY OF KNOWN RESULTS.

Where $\kappa, \lambda$ are infinite cardinals with $\kappa \leq \lambda$, $P^\lambda_\kappa$ denotes the set of all subsets of $[\kappa]^\lambda$ of cardinality $\kappa$. A measure, $\mu$, on $P^\lambda_\kappa$ is a $\{0,1\}$-valued function defined on subsets of $P^\lambda_\kappa$ satisfying

1. $\forall \alpha < \lambda$, $\mu(\{\alpha\}) = 0$ (non-triviality);
2. $\mu(\emptyset) = 0$ and $\mu(P^\lambda_\kappa) = 1$;
3. if $\gamma < \kappa$ and $\{X_\alpha | \alpha < \gamma\}$ is pairwise disjoint, then $\mu(\bigcup X_\alpha) = \sum_{\alpha < \gamma} \mu(X_\alpha)$ ($\kappa$-additivity).

A measure $\mu$ on $P^\lambda_\kappa$ is called normal if, in addition, it satisfies

1. $\forall f : P^\lambda_\kappa \to \lambda$ such that $\mu(\{p \in P^\lambda_\kappa | f(p) \in \alpha\}) = 1$, $\exists X \subseteq P^\lambda_\kappa$ with $\mu(X) = 1$
2. and $\exists \alpha < \lambda$ such that $\forall p \in X$, $f(p) = \alpha$.

$\kappa$ is called $\lambda$-supercompact if there exists a normal measure on $P^\lambda_\kappa$. 


\( \kappa \) is supercompact if it is \( \lambda \)-supercompact \( \forall \lambda \geq \kappa \).

This concept was introduced in [7] because of its analogy with the notion of a measurable cardinal. (A cardinal \( \kappa \) is measurable if there exists a \( \mu : P(\kappa) \to \{0,1\} \) satisfying (1) - (3) above.) Many questions about measures on \( P^\kappa \) are motivated by attempts to generalize results that are known for measurable cardinals.

A measure \( \mu \) on the measurable cardinal \( \kappa \) is called normal if \( \forall f : \kappa \to \kappa \) such that \( \mu(\{ \gamma < \kappa | f(\gamma) < \gamma \}) = 1 \), \( \exists X \subseteq \kappa \) with \( \mu(X) = 1 \) and \( \exists \alpha < \kappa \) such that \( \forall \gamma \in X , f(\gamma) = \alpha \).

The theorem that motivates the partition property is the following one of Rowbottom [1]: if \( \mu \) is a normal measure on a measurable cardinal \( \kappa \) and if \( F : \{ \{ \alpha, \beta \} | \alpha < \beta < \kappa \} \to \{0,1\} \), then \( \exists H \subseteq \kappa \) with \( \mu(H) = 1 \) and \( \exists I \in \{0,1\} \) such that \( F(\{ \alpha, \beta \}) = I \) whenever \( \alpha < \beta \) and \( \alpha, \beta \in H \).

Where \( X \subseteq P^\kappa \), let \( [X]^2 = \{ \{ p, q \} | p, q \in X \& p \neq q \} \). A subset \( X \subseteq P^\kappa \) is called homogeneous for \( F \), where \( F : [P^\kappa]^2 \to \{0,1\} \), if \( F \) is constant on \( \{ \{ p, q \} \in [X]^2 | p \subseteq q \& q \subseteq p \} \). A normal measure \( \mu \) on \( P^\kappa \) is said to have the partition property (which we abbreviate by Part (\( \mu \))) if \( \forall F : [P^\kappa]^2 \to \{0,1\} \), \( \exists X \subseteq P^\kappa \) with \( \mu(X) = 1 \) which is homogeneous for \( F \).

By analogy with Rowbottom's theorem, it was natural to conjecture that every normal measure on \( P^\kappa \) would have the partition property. The following list of theorems summarizes the results that are known.

**THEOREM A** (Menas [4]). If \( \kappa \) is \( 2^{<\kappa} \)-supercompact, then there exist \( 2^{2^\kappa} \)-many normal measures on \( P^\kappa \) with the partition property.

**THEOREM B** (Solovay; published in [4]). If \( \kappa \) is supercompact and \( \lambda > \kappa \) is \( \beta \)-supercompact for some \( \beta \geq \lambda \) (in particular, if \( \lambda \) is measurable), then there is a normal measure on \( P^\lambda \) without the partition property.

**THEOREM C** (Solovay, assuming GCH: Menas, without GCH [4]). If \( \kappa \) is supercompact, then for certain small cardinals (e.g. \( \lambda = \kappa^+ \) or \( \lambda = 2^\kappa \)) every normal measure on \( P^\kappa \) has the partition property.

**THEOREM D** (Kunen [3], details are in [6]). If \( \kappa \) is supercompact, then the least \( \alpha > \kappa \) such that \( P^\kappa \) bears a normal measure without the partition property.
is $\Pi^1_1$-indescribable and inaccessible.

2. **MENAS' PROPERTY $\chi$.**

In his proof of Theorem A, Menas introduced an auxiliary combinatorial property. A measure $\mu$ on $P_\kappa^\lambda$ is said to have property $\chi$ (which we will abbreviate as $\chi(\mu)$) if there exists an $f: \kappa \rightarrow \kappa$ such that

$$\mu(\{p \in P_\kappa^\lambda \mid f(|p \cap \kappa|) = |p| \land \forall \alpha < |p \cap \kappa|, f(\alpha) < |p \cap \kappa|\}) = 1 .$$

His proof of Theorem A can be broken into three lemmas.

**LEMMA A_1.** Part(\mu) iff $\exists X \subseteq P_\kappa^\lambda$ with $\mu(X) = 1$ such that $\forall p, q \in X (p \subseteq q \rightarrow |p| < |q \cap \kappa|)$.

**LEMMA A_2.** If $\kappa$ is $2^{\lambda^\kappa} < \kappa$-supercompact, then there exist $2^{2^{\lambda^\kappa}} < \kappa$-many normal measures on $P_\kappa^\lambda$ with property $\chi$.

**LEMMA A_3.** For any normal measure, $\mu$, on $P_\kappa^\lambda$, $\chi(\mu) \rightarrow \text{Part}(\mu)$.

His proof of Lemma A_3 uses the characterization of Part(\mu) given in Lemma A_1.

3. **RESTRICTION MEASURES, EXTENDIBLE MEASURES, AND ELEMENTARY EMBDDEDS.**

Suppose $\kappa \leq \alpha \leq \beta$ and $g: P_\kappa^{\beta} \rightarrow P_\kappa^{\alpha}$, If $\mu$ is a measure on $P_\kappa^{\beta}$, then $g_\kappa(\mu)$ is the $\{0,1\}$-valued function defined on subsets of $P_\kappa^{\alpha}$ by

$$g_\kappa(\mu)(X) = 1 \text{ iff } \mu(\{p \in P_\kappa^{\beta} \mid g(p) \in X\}) = 1 .$$

For the function $g(p) = p \cap \alpha$, it can be proved that $g_\kappa(\mu)$ is a measure on $P_\kappa^{\alpha}$; it is called the restriction of $\mu$ to $P_\kappa^{\alpha}$ and is denoted by $\mu|\alpha$. In this same situation, if $\nu$ is a measure on $P_\kappa^{\alpha}$, we say that $\nu$ is $\beta$-extendible if there exists a measure $\mu$ on $P_\kappa^{\beta}$ such that $\nu = \mu|\alpha$.

For a normal measure, $\mu$, on $P_\kappa^{\lambda}$, let $M$ denote the transitive collapse of $P_\kappa^{\lambda}/\mu$, the ultrapower of the universe, $V_\kappa^\lambda/\mu$, constructed from $\mu$ and let $j_\mu: V \rightarrow M$ be the corresponding elementary embedding. (See [7] for details).

**THEOREM E ([7]).** Let $\kappa \leq \alpha \leq \beta$; let $\mu$ be a normal measure on $P_\kappa^{\beta}$ and let $\nu = \mu|\alpha$. Then there exists an elementary embedding $k: M_\nu \rightarrow M_\mu$ such that $k \circ j_\nu = j_\mu$.

4. **THEOREMS.**

**THEOREM 1. ([5]).** For any normal measure $\mu$ on $P_\kappa^{\lambda}$, $\chi(\mu)$ iff there exists an $f: \kappa \rightarrow \kappa$ such that $j_\mu(f)(\alpha) = \lambda$.
This alternate characterization of property \( \chi \) in terms of the behaviour of the associated elementary embedding can be used to provide easier proofs of Theorems A and C. It is also crucial to the proof of Theorem 2 which shows that Menas' property \( \chi \) is not equivalent to the partition property.

**THEOREM 2.** ([6]). Let \( \kappa < \eta < \lambda < \text{the least inaccessible cardinal} \) \((2^\kappa)^+ \leq \lambda < \eta\), let \( \kappa \) be \( \lambda \)-supercompact, and let \( \mu \) be any normal measure on \( P_\kappa \lambda \); then there exists a cardinal \( \sigma \in [\kappa, \lambda) \) such that \( \text{Part}(\mu|\sigma) \) and \( \neg \chi(\mu|\sigma) \).

The proof uses Theorem 1 and Theorem E to get \( \neg \chi(\mu|\sigma) \) and Theorem D to get \( \text{Part}(\mu|\sigma) \).

**THEOREM 3.** ([5]). Suppose \( \phi(\alpha, \xi) \) is a formula in \( L_{ZF} \) satisfying

1. \( \phi(\alpha, \xi_1) \land \phi(\alpha, \xi_2) \rightarrow \xi_1 = \xi_2 \),
2. \( \forall \nu < \kappa \), \( \exists \xi < \kappa \) \( \phi(\alpha, \xi) \);

let \( \kappa < \lambda \) and assume that for some \( \beta > \lambda \), there exists a normal measure \( \mu \) on \( P_\kappa \beta \) such that \( M_\mu \models \phi(\kappa, \lambda) \); then \( \chi(\mu|\lambda) \).

In particular, if (i) and (ii) hold for \( \phi(\alpha, \xi) \) and \( \nu \) is a normal measure on \( P_\kappa \lambda \) which is \( \beta \)-extendible to a measure \( \mu \) on \( P_\kappa \beta \) for which \( M_\mu \models \phi(\kappa, \lambda) \), then \( \chi(\nu) \) (and hence \( \text{Part}(\nu) \) by Lemma A.3).

The proof uses Theorem 1 in conjunction with Theorem E.

Theorem 3 can be used to show that certain measures are not extendible. The existence of non-extendible measures on a supercompact cardinal is known. In fact, if \( \kappa \) is supercompact and if \( \mu \) is minimal in the Mitchell ordering of normal measures on \( P_\kappa \lambda \) (see [7] for this definition), it can be shown that \( \mu \) is not \( 2^{\kappa} \)-extendible. Here, we show that Solovay's "glue-together" measures are not extendible.

Where \( \kappa \) is \( \lambda \)-supercompact and \( \lambda \) is measurable, the so called "glue-together" measures are defined as follows: fix a normal measure \( \tau \) on \( \lambda \) and a normal measure \( \nu \) on \( P_\kappa \lambda \), and for \( X \subseteq P_\kappa \lambda \), define \( \nu(X) = \tau(\{\sigma < \lambda \mid \nu(X \cap P_\kappa \sigma) = 1 \}) \); this is a special case of the construction used by Solovay in his proof of Theorem B.

**THEOREM 4.** ([5]). If \( \kappa \) is \( \lambda \)-supercompact where \( \lambda \) is the least measurable cardinal greater than \( \kappa \), then the "glue-together" measures on \( P_\kappa \lambda \) are not \( 2^\lambda \)-extendible.
extendible.

The proof uses Theorem 3 in conjunction with Theorem B and Lemma A_3.

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