THE DUAL OF THE MULTIPLIER ALGEBRA OF PEDERSEN'S IDEAL

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ABSTRACT. It is shown that the dual of the multiplier algebra of Pedersen's ideal is not always spanned by its positive elements.

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1. INTRODUCTION.

In [1] Lazar and Taylor study the multiplier algebra $\Gamma(K)$ of Pedersen's minimal dense ideal K of a C^* -algebra A . Equipped with its canonical strict topology, $\Gamma(K)$ is a locally convex space, and Lazar and Taylor have demonstrated that the dual $\Gamma(K)'$ can be identified with the set of all linear functionals $A \cdot F + G \cdot A$ where $A \in K^+$ and $F, G \in A'$ ([1] 6.1). They have also shown that, under this identification each positive element of $\Gamma(K)'$ is of the form $A \star F \cdot A$ for some $A \in K$ and $F \in A'$ ([1] 6.5). This note answers negatively their question, whether or not $\Gamma(K)'$ is the span of its positive elements.

2. MAIN RESULTS.

Let H denote the Hilbert space $\ell_2(\mathbb{Z})$, $\langle, \rangle$ its inner product, and $\{b_n\}_{n \in \mathbb{Z}}$ its canonical Hilbert basis. For vectors $v, w \in H$, let $v \otimes w^*$ denote the linear operator sending each $x \in H$ to $\langle x, w \rangle v$. Denote by $(B, \star, \|\cdot\|_B)$ the C^* -algebra of all bounded

linear transformations of H . The identity transformation I has a decomposition $P + Q$ where P is the orthogonal projection of H onto $\ell_2(\mathbb{N})$ and Q the orthogonal projection onto $\ell_2(Z/\mathbb{N})$.

Let A be the C^* -algebra of all bounded sequences $A: \mathbb{N} \rightarrow B$ such that

$$\lim_{n \rightarrow \infty} \|P A_n\|_B + \|A_n P\|_B = 0. \quad (2.1)$$

We write $\|\cdot\|$ for the norm on A :

$$\|A\| = \sup_{n \in \mathbb{N}} \|A_n\|_B \quad (\forall A \in A).$$

Let K be the set of all $A \in A$ such that

$$\{n \in \mathbb{N} : \|P A_n\| + \|A_n P\| \neq 0\} \text{ is finite.} \quad (2.2)$$

PROPOSITION. Pedersen's ideal in A is just K .

PROOF. That K is an ideal is trivial.

For each $A \in A$ and $n \in \mathbb{N}$, let $A^{(n)}$ be the element of K defined by

$$A_m^{(n)} = A_m \quad (\forall m = 1, 2, \dots, n), \quad A_k^{(n)} = Q A_k Q \quad (\forall k = n+1, n+2, \dots).$$

Then, for each $A \in A$,

$$\begin{aligned} \overline{\lim}_n \|A - A^{(n)}\| &= \overline{\lim}_n \sup_{k > n} \|A_k - Q A_k Q\|_B \leq \\ \overline{\lim}_n \sup_{k > n} \|P A_k Q\|_B + \|Q A_k P\|_B + \|P A_k P\|_B &= 0 \end{aligned}$$

by (2.1), which proves that K is dense in A .

Since the minimal dense ideal of A contains all positive elements $A \in A$ such that $AB = A$ for some $B \in A^+$, it will suffice in showing K is Pedersen's ideal to demonstrate that K is spanned by elements of this sort. Since K is evidently spanned by its positive elements, it will be sufficient to examine an arbitrary positive element A of K . For such $A \in K^+$, there exists $n \in \mathbb{N}$ such that $P A_m = 0 = A_m P$ for all $m > n$. Let $B \in A^+$ be defined by

$$B_k \equiv I \quad (\forall k = 1, 2, \dots, n), \quad B_m \equiv Q \quad (\forall m = n+1, n+2, \dots).$$

Then, since $A_m = Q A_m Q$ for all $m > n$, $AB = A$. Q.E.D.

Let F be the bounded linear functional on A defined by

$$F(A) \equiv \sum_{n=1}^{\infty} \langle A_n(b_1), b_{-1} \rangle \cdot 2^{-n} \quad (\forall A \in A).$$

Let $D \in K$ be the partial unitary operator defined by

$$D_n \equiv Q \quad (\forall n \in \mathbb{N}).$$

For $G \in A'$ and $A \in A$, the linear functionals $A \cdot G$ and $A \cdot G \cdot A$ are defined by

$$A \cdot G(B) \equiv G(AB) \text{ and } A \cdot G \cdot A(B) \equiv G(ABA) \quad (\forall B \in A).$$

Let S be the linear span of the set of all elements of A' of the form $A \cdot G \cdot A$ such that $A \in K^+$ and $G \in A'$.

THEOREM. The linear functional $D \cdot F$ is not in S .

PROOF. Assume false. Then there exists a finite subset F of K^+ and a map $G|_F \rightarrow A'$ such that

$$D \cdot F = \sum_{A \in F} A \cdot G_A \cdot A. \quad (2.3)$$

Choose $n \in \mathbb{N}$ such that

$$PA_m = A_m P = 0 \quad (\forall A \in F; m = n, n+1, \dots).$$

Let $B \in A$ be defined by

$$B_n \equiv b_{-1} \otimes b_1^* \quad \text{and} \quad B_k \equiv 0 \quad (\forall k \in \mathbb{N} : k \neq n),$$

and note that

$$QB_n = B_n \quad \text{and} \quad QB_n Q = 0.$$

We have

$$D \cdot F(B) = F(DB) = \langle (DB)_n(b_1), b_{-1} \rangle 2^{-n} = 2^{-n} \neq 0. \quad (2.4)$$

For each $A \in F$, we have $A_n = QA_n Q$; therefore, since $(b_{-1} \otimes b_1^*)Q = 0$, it follows that $B_n A_n = 0$. Thus

$$BA = 0 \quad \text{and}$$

$$A \cdot G_A \cdot A(B) = G_A(A \cdot BA) = 0 \quad (\forall A \in F). \quad (2.5)$$

But (2.3), (2.4), and (2.5) are incompatible. Q.E.D.

REFERENCE

- [1] Lazar, A. J., and Taylor, D. C., Multipliers of Pedersen's Ideal, Mem. Am. Math. Soc. 169, (1976).