RESEARCH NOTES

SOME REMARKS ON THE STABILITY ANALYSIS IN ROBE'S THREE BODY PROBLEM

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(Received October 12, 1979)

ABSTRACT. An improved technique is presented for the stability analysis of Robe's 3-body problem which gives more accurate results for the transition curves in the parameter plane than does Robe's paper.

A novel property of the system of differential equations describing the motion is used, which reduces the computer time by more than 50%.

KEY WORDS AND PHRASES. 3-body problem, stability, transition curve, Floquet-theory.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. 85A05

1. INTRODUCTION.

In a recent paper Robe [1] presented a new kind of restricted three body problem, where one body m_1 is a rigid spherical shell, filled with an homogeneous incompressible fluid of density ρ_1 , where a second body m_2 is a mass point outside the shell, and where m_3 is a small solid sphere of density ρ_3 , restricted to move inside the shell, its motion determined by the attraction of m_2 and the buoyancy force due to the fluid ρ_1 .

There exists a solution with m_3 at the center of the shell while m_2 describes a Keplerian orbit around it. Robe investigated the stability of this configuration under the assumption that the mass of m_3 is infinitesimal. The linearized equations

of motion in the neighborhood of this equilibrium are

$$\ddot{x} - 2\dot{y} = \left\{ \frac{1 + 2\mu}{1 + e \cos v} - \frac{K(1 - e^2)^3}{(1 + e \cos v)^4} \right\} x \quad (1.1)$$

$$\ddot{y} + 2\dot{x} = \left\{ \frac{1 - \mu}{1 + e \cos v} - \frac{K(1 - e^2)^3}{(1 + e \cos v)^4} \right\} y \quad (1.2)$$

$$\ddot{z} + z = \left\{ \frac{1 - \mu}{1 + e \cos v} - \frac{K(1 - e^2)^3}{(1 + e \cos v)^4} \right\} z \quad (1.3)$$

where

$$K = \frac{4\pi}{3} \cdot \frac{\rho_1}{(m_1 + m_2)} a^3 \left(1 - \frac{\rho_1}{\rho_3}\right)$$

$$\mu = \frac{m_2}{(m_1^* + m_2)}$$

a = semi-major axis of the Keplerian orbit.

These equations are referred to a coordinate system Oxyz, where O is the center of m_1 , Ox points to m_2 and Oxy is the plane of the Keplerian orbit.

If $\rho_1 = 0$ (shell empty) or $\rho_1 = \rho_3$, then $K = 0$ and the equations of motion become

$$\ddot{x} - 2\dot{y} = (1 + 2\mu) r x \quad (1.4)$$

$$\ddot{y} + 2\dot{x} = (1 - \mu) r y \quad (1.5)$$

$$\ddot{z} + z = (1 - \mu) r z \quad (1.6)$$

with $r = \frac{1}{1 + e \cos v}$.

Equations (1.4) and (1.5) describe the motion in the orbital plane. Robe investigated the stability in the orbital plane by means of the Floquet-theory. However, one can separate the fourth-order system (1.4), (1.5) into two independent second-order systems.

2. THE TRANSFORMATION TO SECOND-ORDER SYSTEMS.

Using $\xi = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\eta = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$,

equations (1.4) and (1.5) can be written

$$\begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} 0 & E \\ rC_0 & 2D \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} \quad (2.1)$$

where

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and

$$C_0 = \begin{bmatrix} 1 + 2\mu & 0 \\ 0 & 1 - \mu \end{bmatrix}$$

Now we make the following transformation (Tschauner [2])

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} E & E \\ P_1 & P_2 \end{bmatrix} \begin{bmatrix} \delta \\ \epsilon \end{bmatrix} \quad (2.2)$$

and obtain

$$\begin{bmatrix} \dot{\delta} \\ \dot{\epsilon} \end{bmatrix} = \frac{1}{P_2 - P_1} \cdot \begin{bmatrix} P_2 P_1 - rC_0 - 2DP_1 + P_1' & P_2^2 - rC_0 - 2DP_2 + P_2' \\ -P_2^2 + rC_0 + 2DP_1 - P_1' & -P_1 P_2 + rC_0 + 2DP_2 - P_2' \end{bmatrix} \begin{bmatrix} \delta \\ \epsilon \end{bmatrix}$$

Making the nondiagonal elements zero, we obtain

$$\begin{bmatrix} \dot{\delta} \\ \dot{\epsilon} \end{bmatrix} = \begin{bmatrix} P_1 & \\ & P_2 \end{bmatrix} \begin{bmatrix} \delta \\ \epsilon \end{bmatrix} \quad (2.3)$$

where P_1 and P_2 are two different solutions of the Riccati equation

$$\dot{P} = 2DP - P^2 + rC_0. \quad (2.4)$$

Using

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix},$$

equation (2.4) becomes

$$\left. \begin{aligned} \dot{p}_{11} &= 2p_{21} - p_{11}^2 - p_{12}p_{21} + r(1 + 2\mu) \\ \dot{p}_{22} &= -2p_{12} - p_{22}^2 - p_{12}p_{21} + r(1 - \mu) \\ \dot{p}_{12} &= 2p_{22} - p_{12}(p_{11} + p_{22}) \\ \dot{p}_{21} &= -2p_{11} - p_{21}(p_{11} + p_{22}) \end{aligned} \right\} \quad (2.5)$$

Now let

$$\left. \begin{aligned} p_{11} &= w + z \\ p_{22} &= w - z \\ p_{12} &= u - v + 1 \\ p_{21} &= u + v - 1 \end{aligned} \right\} \quad (2.6)$$

Then equation (2.5) becomes

$$\left. \begin{aligned} \dot{w} &= -1 - w^2 - z^2 - u^2 + v^2 + r\left(\frac{2+u}{2}\right) \\ \dot{z} &= 2u - 2wz + r\frac{3}{2}\mu \\ \dot{u} &= -2z - 2uw \\ \dot{v} &= -2vw \end{aligned} \right\} \quad (2.7)$$

The last two equations yield

$$\left. \begin{aligned} w &= \frac{v}{2} \left(\frac{\dot{1}}{v} \right) \\ z &= -\frac{v}{2} \left(\frac{\dot{u}}{v} \right) \end{aligned} \right\}$$

If we use $p = \frac{1}{v}$ and $q = \frac{u}{v}$, we get

$$\left. \begin{aligned} w &= \frac{\dot{p}}{2p} \\ z &= -\frac{\dot{q}}{2p} \end{aligned} \right\} \quad (2.8)$$

Substituting (2.8) into the first two equations of (2.7), we have

$$p \cdot \ddot{p} - \frac{1}{2} \dot{p}^2 + [\lambda - r(2 + \mu)] p^2 - 2 = -2(q^2 + \frac{1}{4} \dot{q}^2) \quad (2.9)$$

$$\ddot{q} + 4q = -3r\mu \quad (2.10)$$

Now if we let

$$rp = k_0 + k_1 \cos v, \quad (2.11)$$

we find as a solution for (2.10)

$$q = -\frac{3}{4} \mu k_0 - \mu k_1 \cos v + k_2 \cos^2 v \quad (2.12)$$

If we substitute the solutions (2.11) and (2.12) into (2.9), we obtain (by identification) the values for k_0 , k_1 and k_2 as functions of μ and e .

For k_0 , we obtain

$$k_0 = \pm \frac{4}{c}$$

$$\text{with } c = \sqrt{\frac{4e^4}{(3\mu + 1)^2} + \frac{4e^2\mu}{(3\mu + 1)^2} - (\mu + 3) + \mu(9\mu - 8)} \quad (2.13)$$

This will yield two different solutions, P_1 and P_2 , if $c \neq 0$ and $c^2 > 0$.

So $c(\mu, e) = 0$ will give us (analytically) a transition curve in the μ - e plane, which corresponds to one of the transition curves (IE) that Robe obtained numerically.

This curve can be written as

$$\boxed{\frac{4e^4}{(3\mu + 1)^2} + \frac{4e^2\mu}{(3\mu + 1)^2} - (\mu + 3) + \mu(9\mu - 8) = 0} \quad (2.14)$$

The elements p_{ij} of P_1 and P_2 are

$$p_{11} = \frac{-(2\mu + 1)e \sin v}{(3\mu + 1 + e \cos v)(1 + e \cos v)} \quad (2.15)$$

$$p_{22} = \frac{-(\mu + 1 + 2e \cos v) e \sin v}{(3\mu + 1) e \cos v (1 + e \cos v)} \quad (2.16)$$

$$p_{12} = \frac{(3\mu + 1) \left(1 - \frac{3}{4} \mu \pm \frac{c}{4}\right) + \frac{e^2}{2} + 2e(\mu + 1) \cos v + e^2 \cos 2v}{(3\mu + 1 + e \cos v) (1 + e \cos v)} \quad (2.17)$$

$$p_{21} = \frac{-[(3\mu + 1) \left(1 + \frac{3}{4} \mu \pm \frac{c}{4}\right) + \frac{e^2}{2} + 2(2\mu + 1) e \cos v]}{(3\mu + 1 + e \cos v) (1 + e \cos v)} \quad (2.18)$$

3. STABILITY ANALYSIS.

Now we perform a stability analysis using the Floquet theory, on the two independent second order systems (2.3)

$$\dot{\delta} = P_1 \delta \quad (3.1)$$

$$\dot{\varepsilon} = P_2 \varepsilon \quad (3.2)$$

Both equations admit solutions for which

$$u(v + 2\pi) = s_i u(v) \quad (i = 1, 2)$$

where s_i are the roots of the characteristic equation

$$\det [X^{-1}(v) X(v + 2\pi) - sE] = 0 \quad (3.3)$$

where $X(v)$ is a fundamental solution matrix of (3.1) (or (3.2)). Equation (3.3) can be written

$$s^2 - 2\alpha s + 1 = 0 \quad (3.4)$$

For stable solutions,

$$|\alpha| < 1 \quad (3.5)$$

Thus the transition curves in the $(\mu - e)$ plane, separating stable and non-stable regions, will be given by

$$|\alpha| = 1$$

or

$$s = \pm 1 \quad (3.6)$$

Taking $X(v = 0) = E$, equation (3.3) becomes

$$\det [X(2\pi) - sE] = 0$$

where $X(2\pi) = \begin{bmatrix} \alpha & \beta \\ \gamma & \alpha \end{bmatrix}$ is the monodromy-matrix and $\alpha^2 - \beta\gamma = 1$.

It can be shown (R. Meire and A. Vanderbauwhede [3]) that

$$X(2\pi) = SX^{-1}(\pi)SX(\pi) \quad (3.7)$$

where

$$S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

This implies that we only have to integrate the equations over π instead of 2π .

One can also prove that for $s = \pm 1$, one of the elements of $X(\pi)$ becomes zero which saves an additional 10% of computer time.

4. RESULTS.

We applied this method to the equations of (3.1) and (3.2). Equation (3.1) yielded two transition curves: FM and FE. Equation (3.2) yielded one transition curve HM. All results are given in Fig. 1.

The curve IE is the analytically obtained curve from equation (2.14).

Along

$$\left\{ \begin{array}{lll} \text{FM} & \beta=0 & \text{and } \alpha=-1 \\ \text{FE} & \gamma=0 & \text{and } \alpha=-1 \\ \text{HM} & \beta=\gamma=0 & \text{and } \alpha=1 \end{array} \right.$$

The intersection of the curves FE and IE can be obtained very accurately (Robe was not able to give precise coordinates).

The point E is determined as that point on the curve IE for which the characteristic roots of (3.1) are -1. The coordinates of the interesting points in the μ - e plane are

$$\begin{array}{ll} \text{point F} & \left\{ \begin{array}{l} \mu = 0.928053\dots = \frac{5 + \sqrt{97}}{16} \\ e = 0 \end{array} \right. \\ \text{point I} & \left\{ \begin{array}{l} \mu = 0.8888\dots = \frac{8}{9} \\ e = 0 \end{array} \right. \\ \text{point E} & \left\{ \begin{array}{l} \mu = 0.8596848 \\ e = 0.4531741 \end{array} \right. \end{array}$$

The stable region consists of the shaded area in Fig. 1 and is now determined much

more accurately than in Robe's paper where the fourth-order system (2.1) was used.

Fig. 1 : Stability-regions in the μ - e -plane

ACKNOWLEDGEMENTS.

The author thanks Prof. Dr. P. Dingens and Prof. Dr. H. Steyaert for reading the manuscript and for their valuable suggestions on this work.

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