ON SOME FIXED POINT THEOREMS IN BANACH SPACES

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ABSTRACT. In this paper, some fixed point theorems are proved for multi-mappings as well as a pair of mappings. These extend certain known results due to Kirk, Browder, Kanna, Ćirić and Rhoades.

KEY WORDS AND PHRASES. Normal structure, Multi-mapping, Uniformly convex Banach Space.

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1. INTRODUCTION.

A result of continuing interest in fixed point theory is one due to Kirk [6]. This states that a non-expansive self-mapping of bounded, closed and convex subset possessing normal structure in a reflexive Banach space has a fixed point. The interest in this result has been further enhanced due to simultaneous and independent appearance of results of Browder [2] and Göhde [5] which are essentially special cases of the result of Kirk. Recently Kannan [6] and Ćirić [2] have obtained results in basically the same spirit by suitably modifying the non-expansive condition on the mapping and the condition of normal structure on the underlying set. In this paper we give a fixed point result for multi-mappings (Theorem 2.1) and extend the results of Kannan [6] and Ciric [3] to a pair of mappings (Theorems 3.1 and 3.2). This enables us to establish convergence of Ishikawa iterates (cf. [9]) for a pair of mappings.
2. A FIXED POINT THEOREM FOR MULTI-MAPPINGS.

Let $K$ be a closed, bounded and convex subset of a Banach space $X$. For $x \in X$, let $\delta(x;K)$ denote $\sup \{ \|x-k\| : k \in K\}$ and let $\delta(K)$ denote the diameter of $K$. Recall that a point $x \in K$ is called a non-diametral point of $K$ if $\delta(x;K) < \delta(K)$ and that $K$ is said to have normal structure whenever given any closed bounded convex subset $C$ of $K$ with more than one point, there exists a non-diametral $x \in C$.

It is well-known (cf. [4]) that a compact convex subset of an arbitrary Banach space and a closed, bounded and convex subset of a uniformly convex Banach space have normal structure. With $K$ as before, let $r(K)$ denote the radius of $K$.

Let $M$ denote the Chebyshev centre of $K$: $\inf \{ \delta(x;K) : x \in K\}$ and let $K_c$ denote the Chebyshev centre of $K$.

Theorem 2. Let $K$ be a nonempty weakly compact convex subset of the Banach space $X$. Assume $K$ has normal structure. Let $T:K \to 2^K$ be a mapping satisfying:

for each closed convex subset $F$ of $K$ invariant under $T$, there exists some $\alpha(F), 0 < \alpha(F) < 1$, such that

$$\delta(Tx,Ty) \leq \max \{ \delta(x,F), \alpha(F) \delta(F) \}$$

for each $x, y \in F$.

Then $T$ has a fixed point $x_0$ satisfying $Tx_0 = \{x_0\}$.

Proof. We imitate in parts the proof of Kirk's theorem. Let $\mathcal{F}$ denote the collection of non-empty closed convex subsets $C$ of $K$ that are left invariant by $T$, i.e., $Tc \subseteq C$, where $TC = \cup \{Tc : c \in C\}$. Order $\mathcal{F}$ by set-inclusion. By weak compactness of $K$, we can apply Zorn's lemma to get a minimal element $M$. It suffices to show that $M$ is a singleton. Suppose that $M$ contains more than one element. By the definition of normal structure there exists $x_0 \in M$ such that

$$\sup \{ \|x_0-y\| : y \in M \} = \delta(x_0,M) < \delta(M),$$

Hence $\delta(x_0,M) \leq \alpha_1(M) \delta(M)$ for some $\alpha_1, 0 < \alpha_1 < 1$. 
If $\delta(Tx, Ty) \leq \delta(x, M)$ for all $x, y \in M$, let $M_\delta = \{x \in M : \delta(x, M) \leq \alpha_1 \delta(M)\}$.

Otherwise, by hypothesis there exists $\alpha(M), 0 \leq \alpha(M) < 1$, such that $\delta(Tx, Ty) \leq \alpha \delta(M)$ for some $x, y \in M$.

Let $\beta = \max \{\alpha, \alpha_1\}$ and $M_\delta = \{x \in M : \delta(x, M) \leq \beta \delta(M)\}$.

As $x_0 \in M_\delta$, $M_\delta$ is nonempty. Evidently, $M_\delta$ is convex. Since $x \rightarrow \delta(x, M)$ is continuous, $M_\delta$ is closed.

Let $x \in M_\delta$

$$\delta(Tx, Ty) \leq \max \{\delta(x, M), \alpha \delta(M)\}$$

$$\leq \beta \delta(M) \text{ for } y \in M.$$

Hence $T(M)$ is contained in a closed ball of arbitrary centre in $Tx$ and radius $\beta \delta(M)$. By the minimality of $M$, if $m \in Tx$, then $M \subseteq U(\ m : \beta \delta(M))$ (the closed ball of centre $m$ and radius $\beta \delta(M)$), whence $m \in M_\delta$ and $T(M_\delta) \subseteq M_\delta$. But

$$\delta(M_\delta) \leq \beta \delta(M) < \delta(M)$$

which contradicts the minimality of $M$. Thus $M$ is a singleton and this completes the proof.

**Corollary 2.2.** Let $K$ be a nonempty weakly compact convex subset of the Banach space $X$. Assume $K$ has normal structure. Let $T$ be a mapping of $K$ into itself which satisfies: for each closed convex subset $F$ of $K$ invariant under $T$ there exists some $\alpha(F), 0 \leq \alpha(F) < 1$, such that

$$||Tx-Ty|| \leq \max \{\delta(x, F), \alpha \delta(F)\}$$

for each $x, y \in F$. Then $T$ has a fixed point.

**Corollary 2.3.** Let $K$ be a nonempty weakly compact convex subset of the Banach space $X$. Assume $K$ has normal structure. Let $T$ be a mapping of $K$ into itself which satisfies: for each closed convex subset $F$ of $K$ invariant under $T$ there exists some $\alpha(F), 0 \leq \alpha(F) < 1$, such that

$$||Tx-Ty|| \leq \max \{||x-y||, r(F), \alpha \delta(F)\}$$

for each $x, y \in F$. Then $T$ has a fixed point.

**Remark.** The preceeding results generalize the results of Kirk [7] and Browder [2].
3. **COMMON FIXED POINTS OF MAPPINGS.**

**Theorem 3.1.** Let $K$ be a weakly compact convex subset of the Banach space $X$.

Let $T_1$, $T_2$ be two mappings of $K$ into itself satisfying:

1. $||T_1 x - T_2 y|| \leq \max \{ (||x-T_1 x|| + ||y-T_2 y||)/2$, $||x-T_2 y|| + ||y-T_1 x||)/3$, $||x-y|| + ||x-T_1 x|| + ||y-T_2 y||)/3\}$

for each $x, y \in K$,

2. $T_1 C \subseteq C$ if and only if $T_2 C \subseteq C$ for each closed subset $C$ of $K$;

3. either $\sup_{z \in C} ||z-T_1 z|| \leq \delta(C)/2$,
   or $\sup_{z \in C} ||z-T_2 z|| \leq \delta(C)/2$ holds for each closed convex subset $C$ of $K$ invariant under $T_1$ and $T_2$.

Then there exists a unique common fixed point of $T_1$ and $T_2$.

**Proof.** Let $\mathcal{F}$ denote the family of all non-empty closed convex subsets of $K$, each of which is mapped into itself by $T_1$ and $T_2$. Ordering $\mathcal{F}$ by set-inclusion, by weak compactness of $K$ and Zorn's lemma, we obtain a minimal element $F$ of $K$. Without loss of generality, assume that $z \in F$ if and only if $z \leq (\delta(C))/2$.

Let $x \in F$. Since $(\delta(F))/2 \leq r(f)$, we obtain using (1) that $||T_1 x - T_2 y|| \leq r(F)$. (y \in F). This gives that $T_2 F \subseteq U(T_1 x : r(f)) = U$, whence $T_2 (F \cap U) \subseteq F \cap U$ and by hypotheses (2) $T_1 (F \cap U) \subseteq F \cap U$. By the minimality of $F$, we obtain $F \subseteq U$.

This gives $\delta(T_1 x, F) = r(F)$, whence $T_1 x \in F$. Therefore, $T_1 (F \cap U) \subseteq F \cap U$ and by hypothesis (2) $T_2 (F \cap U) \subseteq F \cap U$. We now show that if $F$ contains more than one element, then $F$ is a proper subset of $F$. Assume the contrary that $F = F$. Since $\delta(x, F) = r(F)$ for each $x \in F$, we obtain $\delta(F) = r(F) = \delta(x, F)$, ($x \in F$). Again from (1), we get

$||T_1 x - T_2 y|| \leq \max \{ (3 \delta(F))/4$, $(\delta(F) + \delta(F))/3$, $(\delta(F) + \delta(F) + \delta(F)/2)/3\}$

$= 5\delta(F)/6$. 


The same argument as before yields \( \delta(T_1x,F) \leq 5\delta(F)/6 < \delta(F) \), which is a contradiction. Consequently, if \( F \) contains more than one element, then \( F \) is a proper subset of \( F \). But this in view of above contradicts the minimality of \( F \). Hence \( F \) contains exactly one element, say, \( x_0 \), whence \( T_1x_0 = x_0 = T_2x_0 \). Assume there exists another element \( y_0 \in K \) such that \( T_1y_0 = y_0 = T_2y_0 \). Then using (1), we obtain

\[
||T_1x_0 - T_2y_0|| \leq \frac{2}{3} ||T_1x_0 - T_2y_0||,
\]

whence

\[x_0 = T_1x_0 = T_2y_0 = y_0.\]

**THEOREM 3.2.** Let \( K \) be a weakly compact convex subset of the Banach space \( X \).

Assume \( K \) has normal structure. Let \( T_1, T_2 \) be mappings of \( K \) into itself satisfying:

1. \[ ||T_1x - T_2y|| \leq \max \left\{ \frac{1}{2} (||x - T_1x|| + ||y - T_2y||), \frac{1}{2} (||x - T_2y|| + ||y - T_1x||), \frac{1}{3} (||x - y|| + ||x - T_1x|| + ||y - T_2y||) \right\}, \]

for each \( x, y \in K \),

2. \[ T_1C \subseteq C \text{ if and only if } T_2C \subseteq C \text{ for each closed convex subset } C \text{ of } K, \]

3. either \[ \sup_{z \in D} ||z - T_1z|| \leq r(D), \]
   or \[ \sup_{z \in D} ||z - T_2z|| \leq r(D) \]

holds for each closed convex subset \( D \) of \( K \) invariant under \( T_1 \) and \( T_2 \). Then there exists a unique common fixed point of \( T_1 \) and \( T_2 \).

**PROOF.** Let \( \mathcal{F} \) be as in Theorem 3.1. Exactly as in Theorem 3.1., \( \mathcal{F} \) has a minimal element \( F \). Without loss of generality, assume that \( \sup_{z \in F} ||z - T_2z|| \leq r(F) \).

Let \( x \in \mathcal{F}_c \). Then using (1) we obtain

\[ ||T_1x - T_2y|| \leq r(F). \]

This gives exactly as in Theorem 3.1 that \( T_1(F) \subseteq F \) and \( T_2(F) \subseteq F \). Since \( K \) has normal structure, one has \( \delta(F) < \delta(F) \) if \( K \) contains more than one element, which contradicts the minimality of \( F \). Thus \( F \) contains precisely one element, which is the unique common fixed point of \( T_1 \) and \( T_2 \) as in Theorem 3.1.

**REMARK.** One can replace condition (1) of Theorem 3.2 by

1. \[ ||T_1x - T_2y|| \leq \max \left\{ ||x - y||, \frac{1}{2} (||x - T_1x|| + ||y - T_2y||), \frac{1}{3} (||x - y|| + ||x - T_1x|| + ||y - T_2y||) \right\}, \]

(\( \mathcal{F} \))
This also yields the existence of a common fixed point of $T_1$ and $T_2$. However, it need not be unique.

**THEOREM 3.3.** Let $K$ be a weakly compact convex subset of the Banach space $X$. Assume $K$ has normal structure. Let $T_1, T_2$ be mappings of $K$ into itself satisfying (2) and (3) of the preceding theorem and,

\[ \| T_1 x - T_2 y \| \leq \max \{ \| x - y \|, \| x - T_1 x \|, \| x - T_2 x \|, \| x - T_2 y \| \} \]

Then there exists a common fixed point of $T_1$ and $T_2$.

The proof of the above theorem is similar to that of Theorem 3.2 and hence it is omitted.

### 4. ISHIKAWA ITERATION FOR COMMON FIXED POINTS

A uniformly convex Banach space is reflexive. A bounded, closed and convex subset of a uniformly convex Banach space is therefore weakly compact; also, it has normal structure. Hence Theorems 2.1, 3.2 and 3.3 can be particularized to such a setting. Rhoades [9] has extended a result of Ćirić (cf. [3], Theorem 2) to a wider class of transformations by using Ishikawa iterative scheme. With a suitable modification of arguments, this extends to a pair of mappings of the type as in Theorem 3.2.

**THEOREM 4.1.** Let $K$ be a non-empty closed bounded and convex subset of a uniformly convex Banach space $X$. Let $T_1, T_2$ be mappings of $K$ into itself satisfying (1), (2) and (3) of Theorem 3.2. Let the sequence $\{x_n\}$ of iterates be defined by

\[
\begin{align*}
  x_0 & \in K, \\
  y_n &= (1 - \beta_n)x_n + \beta_n T_1 x_n, \quad n \geq 0, \\
  x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T_2 y_n, \quad n \geq 0,
\end{align*}
\]

where $\{\alpha_n\}, \{\beta_n\}$ satisfy (i) $0 \leq \alpha_n, \beta_n \leq 1$ for all $n$, (ii) $\sum \alpha_n (1 - \alpha_n) = \infty$, and (iii) $\lim \beta_n = \beta < 1$. Then $\{x_n\}$ converges to the unique common fixed point of $T_1$ and $T_2$.

**PROOF.** The existence of the unique common fixed point of $T_1$ and $T_2$ results from Theorem 3.2. Let the unique common fixed point be $v$. From (1)

\[ \| T_1 x_n - v \| \leq \| x_n - v \| \]
Following exactly the same lines as in the proof of Theorem 1 of [9] we obtain subsequences \( y_{n_k}, x_{n_k} \) of \( y_n, x_n \) respectively such that

\[
\lim_{k} \| x_{n_k} - T_2 y_{n_k} \| = 0
\]

we show that

\[
\lim_{k} \| x_{n_k} - T_1 x_{n_k} \| = 0.
\]

It would be sufficient, with (7), to show that \( \lim_{k} \| T_1 x_{n_k} - T_2 y_{n_k} \| = 0 \).

For any integer \( n \), from

\[
\| T_1 x_n - T_2 y_n \| \leq \left( \| x_n - T_1 x_n \| + \| y_n - T_2 y_n \| \right) / 2,
\]

we obtain

\[
\| T_1 x_n - T_2 y_n \| \leq (2 - \beta_n) \| x_n - T_2 y_n \| / (1 - \beta_n).
\]

It follows from

\[
\| T_1 x_n - T_2 y_n \| \leq \left( \| x_n - y_n \| + \| x_n - T_1 x_n \| + \| y_n - T_2 y_n \| \right) / 3,
\]

that

\[
\| T_1 x_n - T_2 y_n \| \leq (2 - \beta_n) \| x_n - T_2 y_n \| / (2 + \beta_n).
\]

From

\[
\| T_1 x_n - T_2 y_n \| \leq \left( \| x_n - y_n \| + \| x_n - T_1 x_n \| + \| y_n - T_2 y_n \| \right) / 3
\]

we obtain

\[
\| T_1 x_n - T_2 y_n \| \leq \| x_n - T_2 y_n \| / (1 - \beta_n) + \| y_n - T_2 y_n \| / (1 - \beta_n).
\]

From (9) - (11) we obtain

\[
\| T_1 x_n - T_2 y_n \| \leq 2 \| x_n - T_2 y_n \| / (1 - \beta_n).
\]

Therefore,

\[
\| T_1 x_{n_k} - T_2 y_{n_k} \| \leq 2 \| x_{n_k} - T_2 y_{n_k} \| / (1 - \beta_n)
\]

and (7) implies \( \lim_{k} \| T_1 x_{n_k} - T_2 y_{n_k} \| = 0 \),

whence

\[
\lim_{k} \| x_{n_k} - T_1 x_{n_k} \| = 0.
\]

Now let us prove that this implies that
This follows easily from
\[
||x_{n_k} - T_2x_{n_k}|| \leq ||x_{n_k} - T_1x_{n_k}|| + ||T_1x_{n_k} - T_2x_{n_k}||
\leq ||x_{n_k} - T_1x|| + \max(\{||x_{n_k} - T_1x|| + ||x_{n_k} - T_2x||\}/2, \\
(\{||x_{n_k} - T_2x|| + ||x_{n_k} - T_1x||\}/3, \\
(\{||x_{n_k} - x_{n_k}|| + ||x_{n_k} - T_1x|| + ||x_{n_k} - T_2x||\}/3).
\]
which tends to 0 as \( k \to \infty \) since
\[
||x_{n_k} - T_1x_{n_k}|| \to 0 \text{ as } k \to \infty.
\]
Also
\[
||T_1x_{n_k} - T_1x_{n_k}|| \leq ||T_1x_{n_k} - T_2x_{n_k}|| + ||T_2x_{n_k} - T_1x_{n_k}||
\]
From (1) of Theorem 3.2,
\[
||T_1x_{n_k} - T_2x_{n_k}|| \leq \max\{||x_{n_k} - T_1x_{n_k}|| + ||x_{n_k} - T_2x_{n_k}||\}/2, \\
(\{||x_{n_k} - T_2x_{n_k}|| + ||x_{n_k} - T_1x_{n_k}||\}/3, \\
(\{||x_{n_k} - x_{n_k}|| + ||x_{n_k} - T_1x_{n_k}|| + ||x_{n_k} - T_2x_{n_k}||\}/3).
\]
If
\[
||T_1x_{n_k} - T_2x_{n_k}|| \leq ||T_1x_{n_k} - T_2x_{n_k}|| + ||x_{n_k} - T_1x_{n_k}||/3, \text{ then}
3 ||T_1x_{n_k} - T_2x_{n_k}|| \leq ||x_{n_k} - T_2x_{n_k}|| + ||T_1x_{n_k} - T_2x_{n_k}||
+ ||x_{n_k} - T_2x_{n_k}|| + ||T_1x_{n_k} - T_1x_{n_k}||
\]
which implies
\[
(11) \quad ||T_1x_{n_k} - T_2x_{n_k}|| \leq ||x_{n_k} - T_1x_{n_k}|| + ||x_{n_k} - T_2x_{n_k}||.
\]
If
\[
||T_1x_{n_k} - T_2x_{n_k}|| \leq ||x_{n_k} - x_{n_k}|| + ||x_{n_k} - T_1x_{n_k}|| + ||x_{n_k} - T_2x_{n_k}||/3,
\]
it follows, in a similar manner, that (11) holds. Therefore, in all cases, (11) is satisfied.
Therefore,

\[ |T_1 x_{n_k} - T_1 x_k| \leq |T_1 x_{n_k} - x_k| + |x_n - T_2 x_{n_k}| + |x_n - T_1 x_{n_k}| + |x_n - T_2 x_{n_k}|, \]

which tends to 0 as \( k \to \infty \). Therefore \( \{T_1 x_{n_k}\} \) is a Cauchy sequence and hence it converges, say, to \( u \). Consequently

\[ \lim x_{n_k} = \lim T_1 x_{n_k} = u. \]

Also,

\[ |u - T_2 u| \leq |u - x_{n_k}| + |x_n - T_1 x_{n_k}| + |T_1 x_{n_k} - T_2 u| \leq |u - x_{n_k}| + |x_n - T_1 x_{n_k}| + \max \left\{ \frac{|x_{n_k} - T_2 u|}{3}, \frac{|x_{n_k} - u|}{3}, \frac{|x_{n_k} - T_1 x_{n_k}|}{3}, \frac{|u - T_2 u|}{3} \right\}. \]

Taking the limit as \( k \to \infty \), we obtain \( |u - T_2 u| = 0 \). Therefore, \( u = T_2 u \).

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