BAZILEVIC FUNCTIONS OF TYPE β

K. INAYAT NOOR

Mathematics Department
Islamia University
Bahawalpur, Pakistan

(Received March 12, 1980 and in revised form December 7, 1980)

ABSTRACT. In this paper, a new coefficient result for the Bazilevič functions of type β is obtained.

KEY WORDS AND PHRASES. Bazilevič functions, Starlike, Coefficients, Close-to-convex, Univalent functions.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. 30C45, 30C50.

1. INTRODUCTION.

Let S denote the class of functions f which are analytic and univalent in $E = \{z: |z| < 1\}$ and which satisfy $f(0) = 0$ and $f'(0) = 1$. Let S^* be the class consisting of starlike functions. Bazilevič [1] introduced a class of analytic functions f defined by the following relation. For $z \in E$, let

$$f(z) = \left\{ \frac{\beta}{1+a^2} \int_0^z (H(\xi) - ai) \xi^{\frac{-\beta ai}{1+a^2} - 1} \frac{\beta/1+a^2}{g} (\xi) d\xi \right\}^{\frac{1+ai}{\beta}}, \quad (1.1)$$

where a is real, $\beta > 0$, $\operatorname{Re} H(z) > 0$ and $g \in S^*$. Such functions, he showed, are univalent [1]. With $a = 0$ in (1.1), we have [2] for $z \in E$,

$$\operatorname{Re} \frac{zf'(z)}{f^{1-\beta}(z)g^\beta(z)} > 0. \quad (1.2)$$

We shall denote this class of functions by $B(\beta)$. We notice that, if $\beta = 1$ in (1.2), we have the class of close-to-convex functions first introduced by Kaplan [3].

2. MAIN RESULTS.

Denote $M(r, f) = \max_{|z=r|} |f(z)|$, $0 \leq r < 1$ and $M(r, f) \leq (1-r)^{-\alpha}$, $0 \leq \alpha \leq 2$.

Thomas [2] has proved that $n|a_n| \leq K(\alpha, \beta)n^\alpha$. We improve his result as follows:

THEOREM 1. Let $f \in B(\beta)$, for $0 < \beta \leq 1$, and be given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$

Then, for $n \geq 2$,

$$n|a_n| \leq A(\beta)M\left(\frac{2n-1}{2n}, f\right),$$

where $A(\beta)$ is a constant depending only upon β .

PROOF. From (1.2), we can write

$$zf'(z) = f^{1-\beta}(z) g^\beta(z) h(z), \operatorname{Re} h(z) > 0; g \in S^* . \quad (2.3)$$

Thus,

$$\begin{aligned} (zf'(z))' &= (1-\beta) f^{-\beta}(z) f'(z) g^\beta(z) h(z) + \beta f^{1-\beta}(z) g^{\beta-1}(z) g'(z) h(z) \\ &\quad + f^{1-\beta}(z) g^\beta(z) h'(z) . \end{aligned} \quad (2.4)$$

Since, with $z = re^{i\theta}$, Cauchy's theorem gives

$$n^2 a_n = \frac{1}{2\pi r^n} \int_0^{2\pi} z(zf'(z))' e^{-in\theta} d\theta ,$$

we have from (2.4),

$$\begin{aligned} n^2 |a_n| &\leq \frac{1}{2\pi r^n} \left\{ (1-\beta) \int_0^{2\pi} |zf'(z) f^{-\beta}(z) g^\beta(z) h(z)| d\theta \right. \\ &\quad + \beta \int_0^{2\pi} |zg'(z) f^{1-\beta}(z) g^{\beta-1}(z) h(z)| d\theta \\ &\quad \left. + \int_0^{2\pi} |f^{1-\beta}(z) g^\beta(z) zh'(z)| d\theta \right\} \\ &= \frac{1}{r^n} |I_1 + I_2 + I_3|, \text{ say.} \end{aligned}$$

Now,

$$\begin{aligned} I_1 &= \frac{(1-\beta)}{2\pi} \int_0^{2\pi} |zf'(z) f^{-\beta}(z) g^\beta(z) h(z)| d\theta \\ &= \frac{(1-\beta)}{2\pi} \int_0^{2\pi} |f'(z)|^2 |f(z)|^{-1} d\theta, \text{ using (2.3).} \end{aligned}$$

In order to estimate this integral, we use the following well-known result [4, p. 46].

Suppose that f is a really mean p -valent in E and that $\frac{1}{2} \leq r < 1$, $0 < \lambda \leq 2$. Then there exists ρ such that $2r - 1 \leq \rho \leq r$ and

$$\frac{1}{2\pi} \int_0^{2\pi} |f'(\rho e^{i\theta})|^2 |f(\rho e^{i\theta})|^{-\lambda-2} d\theta \leq \frac{4PM(r,f)^\lambda}{\lambda(1-r)} . \quad (2.5)$$

With $\rho = 1$ and $\lambda = 1$ in (2.5), we have

$$\frac{1}{2\pi} \int_0^{2\pi} |f'(\rho e^{i\theta})|^2 |f(\rho e^{i\theta})|^{-1} d\theta \leq \frac{4M(r,f)}{(1-r)} .$$

Since $r < \frac{1+\rho}{2}$ and $M(r,f)$ is an increasing function, $M(r,f) \leq M(\frac{1+\rho}{2}, f)$.

Also,

$$\frac{1}{1-r} \leq \frac{2}{1-\rho} \quad \text{since } 2r - 1 \leq \rho \leq r .$$

Thus

$$I_1 \leq 8(1-\beta) \frac{M(\frac{1+\rho}{2}, f)}{(1-\rho)} .$$

Choosing $\rho = 1 - \frac{1}{n}$, we obtain for $n \geq 2$, see [5, p. 238, 240],

$$I_1 \leq 8(1-\beta) M\left(\frac{2n-1}{2n}; f\right) . \quad n .$$

For $z = re^{i\theta}$,

$$\begin{aligned} I_2 &= \frac{\beta}{2\pi} \int_0^{2\pi} |zg'(z)f^{1-\beta}(z)g^{\beta-1}(z)h(z)| d\theta \\ &= \frac{\beta r}{2\pi} \int_0^{2\pi} |f'(z)\phi(z)| d\theta, \text{ where } \phi(z)g(z) = zg'(z); \operatorname{Re} \phi(z) > 0. \end{aligned}$$

Applying the Schwarz inequality, we have

$$I_2 \leq \frac{\beta r}{2\pi} \left(\int_0^{2\pi} |f'(z)|^2 d\theta \right)^{\frac{1}{2}} \left(\int_0^{2\pi} |\phi(z)|^2 d\theta \right)^{\frac{1}{2}} .$$

Now

$$\frac{1}{2\pi} \int_0^{2\pi} |f'(z)|^2 d\theta = \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2} \leq \sum_{n=1}^{\infty} n |a_n|^2 r^n, \quad \max n r^{n-2} .$$

Since the function $\log(nr^n)$ has a maximum at a point $n_0 = \frac{1}{\log \frac{1}{r}}$, we have

$$\log nr^n \leq \log n_0 r^{n_0} = \log \frac{1}{e \log \frac{1}{r}} ;$$

$$\text{i.e., } nr^{n-2} \leq \frac{1}{er^2 \log \frac{1}{r}} \leq \frac{1}{er^2(1-r)} . \quad (2.6)$$

$$\text{Also, it is well-known [6, p. 42] that } M(r, f) \leq \frac{4}{r} M(r^2, f) , \quad (2.7)$$

and that the area principal for univalent functions gives

$$A(r, f) \leq \pi M^2(r, f), \text{ (see [7, p. 215])}. \quad (2.8)$$

Using (2.6), (2.7) and (2.8), we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} |f'(z)|^2 d\theta \leq \frac{16}{e} \frac{M(r, f)^2}{r^3(1-r)} . \quad (2.9)$$

Since $\operatorname{Re} \phi(z) > 0$,

$$\phi(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t) = 1 + \sum_{n=1}^{\infty} c_n z^n \text{ with } |c_n| \leq 2; \quad \text{see [4].}$$

Thus

$$\frac{1}{2\pi} \int_0^{2\pi} |\phi(z)|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} \leq 1 + 4 \sum_{n=1}^{\infty} r^{2n} = \frac{1 + 3r^2}{1 - r^2} \quad (2.10)$$

From (2.9) and (2.10), we have

$$I_2 \leq \frac{4\beta}{\sqrt{er}} \frac{M(r, f)}{(1-r)} \left[\frac{1 + 3r^2}{1 + r} \right]^{\frac{1}{2}} \leq \frac{4\sqrt{2}\beta}{\sqrt{er}} \frac{M(r, f)}{(1-r)} .$$

Since $r \leq \frac{1+r}{2}$ and $M(r, f)$ is an increasing function, we have for $r = 1 - \frac{1}{n}$ and

$n \geq 2$

$$\begin{aligned} I_2 &\leq \frac{4\sqrt{2}\beta}{\sqrt{er}} M\left(\frac{2n-1}{2n}, f\right) \cdot n \\ &\leq \frac{8\beta}{\sqrt{e}} M\left(\frac{2n-1}{2n}, f\right) \cdot n \end{aligned}$$

Finally, since $0 < \beta \leq 1$, $z = re^{i\theta}$ and

$$\begin{aligned} I_3 &= \frac{1}{2\pi} \int_0^{2\pi} |f^{1-\beta}(z) g^\beta(z) zh'(z)| d\theta \\ &\leq M^{1-\beta}(r, f) \cdot \frac{1}{2\pi} \int_0^{2\pi} |g^\beta(z) zh'(z)| d\theta \\ &\leq M^{1-\beta}(r, f) \frac{2r}{1-r^2} \frac{1}{2\pi} \int_0^{2\pi} |g^\beta(z)| \operatorname{Re} h(z) d\theta . \end{aligned}$$

Since it is known [8] that $|zh'(z)| \leq \frac{2r \operatorname{Re} h(z)}{1-r^2}$, then by (3)

$$I_3 \leq M^{1-\beta}(r, f) \frac{2r}{1-r^2} \operatorname{Re} \left\{ \frac{1}{2\pi} \int_0^{2\pi} f^{\beta-1}(z) f'(z) e^{-i \arg g^\beta(z)} d\theta \right\}$$

Integrating by parts and using the fact that g is starlike, we obtain

$$\begin{aligned} I_3 &\leq M^{1-\beta}(r, f) \cdot \beta \frac{M^\beta(r, f)}{1-r} \\ &\leq \beta \frac{M\left(\frac{1+r}{2}, f\right)}{1-r} \\ &= \beta M\left(\frac{2n-1}{2n}, f\right), \text{ n, on choosing } r = 1 - \frac{1}{n}, n \geq 2. \end{aligned}$$

Thus, for $n \geq 2$

$$n|a_n| \leq e \left\{ 8((1-\beta) + \frac{8\beta}{\sqrt{e}} + \beta) \right\} M\left(\frac{2n-1}{2n}, f\right), \text{ (see [4, p. 45]).}$$

This completes the proof.

REFERENCES

1. Bazilevič, I. E. On a case of integrability in quadratures of the Loewner-Vufarev equation, Mat. Sb. 37 (1955), 471-476, Russia.
2. Thomas, D. K. On Bazilevič functions, Trans. Amer. Math. Soc. 132 (1968), 353-361.
3. Kaplan, W. Close-to-convex schlicht functions, Michigan Math. J. (1952), 169-185.
4. Hayman, W. K. Multivalent functions, Cambridge, (1967).
5. Hayman, W. K. On successive coefficients of univalent functions, J. London Math. Soc. 38 (1963), 228-243.
6. Thomas, D. K. Starlike and convex functions, Ph.D. Thesis, University of London, 1967.
7. Pommeranke, Ch. On starlike and convex functions, J. London Math. Soc. 37 (1962), 209-224.
8. Caratheodory, C. Conformal representation, Cambridge, 1932.