A THEOREM ON "LOCALIZED" SELF-ADJOINTNESS OF SCHRÖDINGER OPERATORS WITH $L^1_{\text{loc}}$-POTENTIALS

HANS L. CYCON
New York University
Courant Institute of Mathematical Sciences
251 Mercer Street
New York, New York 10012

(Received December 31, 1981)

ABSTRACT. We prove a result which concludes the self-adjointness of a Schrödinger operator from the self-adjointness of the associated "localized" Schrödinger operators having $L^1_{\text{loc}}$-Potentials.

KEY WORDS AND PHRASES. Schrödinger operators, self-adjointness.

1980 SUBJECT CLASSIFICATION CODES. 35J10, 47B25, 47A55.

1. INTRODUCTION.

In 1978, Simader [1] proved a result which concludes the self-adjointness of a Schrödinger operator from the self-adjointness of the associated "localized" Schrödinger operators. A similar result was given by Brezis [2] in 1979 which seems to be slightly more general than [1]. Both papers deal with Schrödinger operators having $L^2_{\text{loc}}$-potentials.

In this paper, we give an analogous result to [2] for Schrödinger operators with $L^1_{\text{loc}}$-potentials and show the common structure of [1] and [2]. In the proof, we use arguments due to Kato [3] and Simader [2], which are based on quadratic form methods.

We first give some notations (compare [4]). If $t$ is a semi-bounded quadratic form with lower bound $\alpha$, we denote the inner product associated with $t$ by $(u,v)_t = t[u,v] + (1 - \alpha)(u,v)$, for $u,v$ in the form domain $Q(t)$ of $t$. The associated norm will be denoted by $||\cdot||_t$. $t$ is closed if $Q(t)$ together with $(\cdot,\cdot)_t$ is a Hilbert
space. Recall the one-to-one correspondence between semibounded quadratic forms and semibounded self-adjoint operators. If T is a self-adjoint semibounded operator, the domain of the closed form associated with T will be denoted by Q(T) and the form by \( \langle u, v \rangle \mapsto (Tu/v) \) for \( u, v \in Q(T) \). The associated norm will be called the form norm of T. We will always write Q(T) for the Hilbert space of the associated form if the inner product is clear. A set which is dense in the Hilbert space Q(T) will be called a form core of T.

Let \( q \) be a real-valued function on \( \mathbb{R}^n \) and assume
\[
eu{I} (\mathbb{R}^n)_{\text{loc}}
\]
and
\[
\quad (c) \quad 
\]
where the sum in (1.1) is taken in the distributional sense. Then we define a "maximal" operator in \( L^2(\mathbb{R}^n) \) associated with L such that
\[
T_{\text{max}} u := L u
\]
with
\[
D(T_{\text{max}}) := \{ u \in D(L) : Lu (\mathbb{R}^n)_{\max} \}
\]
Consider the quadratic form associated with L
\[
t[w, v] := \int \nabla \cdot L v, \quad w, v \in C^\infty(\mathbb{R}^n).
\]
If we assume
\[
t \text{is bounded from below and closable (without loss of generality } t \equiv 0),
\]
then there exists a semibounded self-adjoint operator \( T_F \) associated with the closure of t. Note that for \( q \in L^2_{\text{loc}}(\mathbb{R}^n) \), \( T_F \) coincides with the Friedrichs extension of \( T_{\text{min}} := T_{\text{max}}|_{C^\infty_o(\mathbb{R}^n)} \); see [3]. \( Q(T_F) \) is then the closure of \( C^\infty_o(\mathbb{R}^n) \) in the sense of the norm \( || \cdot ||_t \) associated with the inner product (\( w, v \),
\[
\quad (i.2) 
\]
\[
\quad (i.3) 
\]
\[
\quad (C_2) 
\]
\[
\quad (1.4) 
\]
Now consider \( \phi \in C^\infty_o(\mathbb{R}^n) \) with \( 0 \leq \phi \leq 1 \) such that \( \phi(x) = 1 \) for \( |x| \leq \frac{1}{2} \) and \( \phi(x) = 0 \) for \( |x| \geq 1 \).
For \( k \in \mathbb{N} \), let
\[
\phi_k(x) := \phi(\frac{x}{k}.
\]
We now assume, for any k, there exists a "localized" operator associated with L; i.e., for k \in \mathbb{N} there exist a q_k \in L_{\text{loc}}^1(\mathbb{R}^n) and a L_k such that

1. \quad L_k u := \Delta u + q_k u \quad (C_3)

with \( D(L_k) := \{ u \in L^2(\mathbb{R}^n) / q_k u \in L_{\text{loc}}^1(\mathbb{R}^n) \} \)

and

2. \quad q_k \phi_k u = q_k u \quad \text{for} \quad u \in D(L).

We define also a "maximal" operator in \( L^2(\mathbb{R}^n) \) associated with L_k; i.e., for k \in \mathbb{N},

\[
T_k u := L_k u
\]

with

\[
D(T_k) := \{ u \in D(L_k) / \text{q_k u} \in L^2(\mathbb{R}^n) \}.
\]

Note, that (C_3) is not really a restriction; see Corollary 1 and Corollary 2.

Denote \( q_k^+ := \max \{ q_k, 0 \}, q_k^- := \max \{ -q_k, 0 \}, q^+ := \max \{ q, 0 \}, q^- := \max \{ -q, 0 \}. \)

2. **MAIN RESULTS.**

**THEOREM.** Let k \in \mathbb{N}. Assume (C_1), (C_2), and (C_3) and define \( T_{\text{max}} \) and \( T_k \) as in (1.2) and (1.6). If we assume additionally,

\[ T_k \text{ is self-adjoint}; \] \hspace{1cm} (C_4)

and

\[ C_0^\infty(\mathbb{R}^n) \text{ is a form core of } T_k \text{ and there exists a } c_k > 0 \] \hspace{1cm} (C_5)

such that

\[ (-\Delta w, w) + (q_k^+ w, w) \leq c_k [(T_k w, w) + \| w \|^2], \quad w \in C_0^\infty(\mathbb{R}^n), \]

(2.1)

then \( T_{\text{max}} \) is self-adjoint.

**PROOF.** First we note that, by (C_5), \( T_k \) is bounded from below by \(-1\). Thus \( Q(T_k) \) is well defined.

Now we proceed in 5 steps.

Step 1. We show that for k \in \mathbb{N}, u \in D(T_{\text{max}}) implies \( \phi_k u \in Q(T_k) \), and thus, by (C_3), \( \phi_k u \in H^1(\mathbb{R}^n) \cap Q(q_k^+) \) and \( q_k u \in L_{\text{loc}}^1(\mathbb{R}^n) \) (making use of the semiboundedness of \( T_k \)).

By \( H^1(\mathbb{R}^n) \), we denote the closure of \( C_0^\infty(\mathbb{R}^n) \) in the usual Sobolev norm

\[ \| u \|_{H^1} := (\| \nabla u \|^2 + \| u \|^2)^{1/2} \].

We have the continuous inclusions (compare Kato [3]),

\[ D(T_k) \subset Q(T_k) \subset H^1(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \subset H^{-1}(\mathbb{R}^n) \subset Q(T) \ast. \]
By $H^{-1}(\mathbb{R}^n)$ and $Q(T_k)^*$, we denote the antidual spaces of $H^1(\mathbb{R}^n)$ and $Q(T_k)$. $T_k + 2$ maps $D(T_k)$ onto $L^2(\mathbb{R}^n)$ and it is well known (see [4]) that this can be extended to a bicontinuous map $T_k^* + 2$ from $Q(T_k)$ onto $Q(T_k)^*$. Actually, $T_k^* + 2$ is a restriction of $L_k + 2$ to $Q(T_k)$ since, by (2.1) and the semiboundedness of $T_k$, $v \in Q(T_k)$ implies $q_k v \in L^1_{loc}(\mathbb{R}^n)$. Now let $u \in D(T_{max})$. Using (C3), we get in the distributional sense

$$L_k \phi_k u = \phi_k T_{max} u - 2 \nabla \phi_k \nabla u - (\Delta \phi_k) u.$$  \hfill (2.2)

Since $\nabla \phi_k u \in H^{-1}(\mathbb{R}^n)$ and all other terms on the right hand side of (2.2) are in $L^2(\mathbb{R}^n)$, we have

$$L_k \phi_k u \in H^{-1}(\mathbb{R}^n) \subset Q(T_k)^*.$$  

Since $T_k^* + 2$ is bijective, we conclude in the same way as Kato [3, Lemma 2] that $\phi_k u \in Q(T_k)$.

Step 2. We show that, for $k \in \mathbb{N}$, $u \in D(T_{max})$ implies $\phi_k u \in Q(T_P)$.

Let $u \in D(T_{max})$. From Step 1, we know $\phi_k u \in H^1(\mathbb{R}^n) \cap Q(q_k^+)$. Then, because of (C3), we also have

$$\phi_k u \in Q(q^+).$$

From a theorem due to Simon [5, Theorem 2.1] (see also [6] for generalizations), we know that $C_0^\infty(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n) \cap Q(q^+)$ in the sense of the norm

$$||w||_{t_k^+} := ||w||^2 + (q^+ w/w) + ||w||^2_2^{1/2}, \ w \in H^1(\mathbb{R}^n) \cap Q(q^+).$$

Therefore, we can find a sequence $\{v_n\}_{n \in \mathbb{N}}$ in $C_0^\infty(\mathbb{R}^n)$ such that

$$||v_n - \phi_k u||_{t_k^+} \rightarrow 0 \ (n \rightarrow \infty).$$  \hfill (2.3)

Then, because of (1.4), we have

$$\phi_k u \in Q(q^-) \quad \text{and} \quad (q^-(v_n - \phi_k u)/(v_n - \phi_k u)) \rightarrow 0 \ (n \rightarrow \infty).$$  \hfill (2.4)

(2.3) and (2.4) imply $\phi_k u \in Q(T_P)$.

Step 3. We show that, for $k \in \mathbb{N}$, $v \in Q(T_k)$ implies $\phi_k v \in Q(T_k) \cap Q(T_P)$ and $u \in Q(T_P)$ implies $\phi_k u \in Q(T_k)$. $\phi_k$ implies $\phi_k v \in Q(T_k) \cap Q(T_P)$ and $u \in Q(T_P)$ implies $\phi_k u \in Q(T_k)$. (2.5)

Let $v \in Q(T_k)$. Then, because of (C3), there exists a sequence $\{v_n\}_{n \in \mathbb{N}}$ in $C_0^\infty(\mathbb{R}^n)$ such that

$$||v_n - v||_{t_k} \rightarrow 0 \ (n \rightarrow \infty),$$  \hfill (2.6)

where $||*||_{t_k}$ denotes the form of $T_k^*$. 
For \( \alpha_k := 1 + \sup |\nabla \phi_k| \), we have
\[
|\nabla \phi_k (v_n - v)| \leq \alpha_k \left( |\nabla (v_n - v)| + |v_n - v| \right)
\]
and
\[
\int q_k^+ |\phi_k (v_n - v)|^2 \leq \int q_k^+ |v_n - v|^2 ;
\]
because of the semiboundedness of \( T_k \), we have
\[
(q_k^- \phi_k (v_n - v)/\phi_k (v_n - v)) \leq |\nabla \phi_k (v_n - v)|^2 + \int q_k^+ |\phi_k (v_n - v)|^2 + |\phi_k (v_n - v)|^2 .
\]
(2.9), together with (2.6), (2.7) and (2.8), yields
\[
\phi_k v \in Q(T_k)
\]
and
\[
|\phi_k v_n - \phi_k v|_{t_k} \longrightarrow 0 \quad (n \to \infty).
\]
Since, by (C3), we have
\[
|\phi_k v_n|_{t_k}^2 = |\phi_k v_n|_{t_k}^2 - |\phi_k v_n|_{t_k}^2 \quad (n \in \mathbb{N}).
\]
(\( \| \cdot \|_{t_k} \) denotes the form norm of \( T_F \)).
We can conclude
\[
|\phi_k (v_n - v_m)|_{t_k} \longrightarrow 0 \quad (n,m \to \infty)
\]
and thus
\[
\phi_k v \in Q(T_F).
\]
(2.10) and (2.11) prove the first part of Step 3.

Now, let \( u \in D(T_F) \) and \( v \in Q(T_k) \). Then \( \phi_k v \in Q(T_k) \cap Q(T_F) \) as proved above
and there exist sequences \( \{u_j\}_{j \in \mathbb{N}} \) and \( \{v_m\}_{m \in \mathbb{N}} \) in \( C_c^\infty (\mathbb{R}^n) \) such that
\[
|u_j - u|_{t} \longrightarrow 0 \quad \text{and} \quad |v_m - v|_{t_k} \longrightarrow 0 \quad (j,m \rightarrow \infty).
\]
Thus,
\[
(T_F u_j, \phi_k v) = \lim_{j,m \to \infty} (T_F u_j, \phi_k v_m) = \lim_{j,m \to \infty} (L_j, \phi_k v_m).
\]
Using (C3), we have
\[
(L_j, \phi_k v_m) = (L_k \phi_k u_j, v_m) - 2(u_j, \nabla \phi_k v_m) - (u_j, v_m \Delta \phi_k).
\]
(2.12) and (2.13) yields, for a suitable constant \( \gamma \in \mathbb{R} \),
\[
\lim_{j \to \infty} (\phi_k u_j, v) = \lim_{j \to \infty} (T_k \phi_k u_j, v) = (T_F u, \phi_k v) + 2(u, \nabla \phi_k v) + \gamma (u, v).
\]
Thus the limit of \( \{\phi_k u_j\}_{j \in \mathbb{N}} \) exists weakly in the Hilbert space \( Q(T_k) \) and since
\[
|\phi_k u_j - \phi_k u| \longrightarrow 0 \quad (j \rightarrow \infty),
\]
we conclude
\[ \phi_k u \in Q(T_k), \]
which proves the second part of Step 3.

Step 4. We show \( T_F \leq T_{\text{max}} \).

Let \( u \in D(T_F) \). Then, for \( k \in \mathbb{N} \) from Step 3, we know \( \phi_k u \in Q(T_k) \) and therefore, by \((C_5)\),
\[ \phi_k u \in H^1(\mathbb{R}^n) \cap Q(q_k^+). \]
As in Step 1, we conclude that
\[ qu \in L^1_{\text{loc}}(\mathbb{R}^n). \]
Thus \( u \in D(L) \) and, from
\[ T_F u = Lu \in L^2(\mathbb{R}^n), \]
we have
\[ u \in D(T_{\text{max}}) \text{ and } T_F u = T_{\text{max}} u. \]

Step 5. We show \( T_F = T_{\text{max}} \).

In view of Step 4, we have to show
\[ D(T_{\text{max}}) \subseteq D(T_F). \]

Let \( v \in D(T_{\text{max}}) \) and
\[ v' := (T_f + 1)^{-1} (T_{\text{max}} + 1)v. \]
Thus, \( v' \in D(T_{\text{max}}) \) by Step 4 and
\[ (T_{\text{max}} + 1)v = (T_F + 1)v' = (T_{\text{max}} + 1)v'. \]
With
\[ u := v - v' \in D(T_{\text{max}}), \]
we conclude \((T_{\text{max}} + 1)u = 0\) and therefore
\[ ((T_{\text{max}} + 1)u, w) = 0 \quad \text{for } w \in C_0(\mathbb{R}^n). \quad (2.14) \]
We will show that \((2.14)\) implies \( u = 0 \); then, Step 5 will be proven.

We argue in the following as Simander does in [1]. Since \( T_{\text{max}} \) is a real operator, we may assume \( u \) to be real-valued. From Step 1, we know that \( \phi_k u \in Q(T_k) \) and thus, by \((C_3)\) and the semiboundedness of \( T_k \),
\[ \phi_k u \in H^1(\mathbb{R}^n) \cap Q(q^+) \cap Q(q^-). \]
If we replace \( w \) in \((2.14)\) by \( \phi_k^2 w \), we get, after some partial integrations,
\begin{align}
&\left(\nabla \varphi_k u, \nabla \varphi_k w\right) + \left(q^+ \varphi_k u / \varphi_k k u\right) - \left(q^- \varphi_k u / \varphi_k k u\right) + \left(\varphi_k u, \varphi_k w\right) = \\
&\left((\nabla \varphi_k)^2 u, w\right) - \left((u \nabla \varphi_k - \varphi_k \nabla u, \varphi_k \varphi_k \varphi_k)\right). \quad (2.15)
\end{align}

Since
\[ u \in H^1_{\text{loc}}(\mathbb{R}^n) \quad \text{and} \quad q^+ \left| \varphi_k u \right| \in L^1(\mathbb{R}^n), \]
we can, by using an approximation, replace \( w \) in (2.15) by \( u^{(m)} \in H^1_{\text{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \), defined by
\[ u^{(m)} := \begin{cases} 
  u(x) & \text{for } |u(x)| \leq m \\
  m \text{ sign}(u(x)) & \text{for } |u(x)| > m
\end{cases} \quad \text{for } m \in \mathbb{N}.
\]

Then, the limits of both sides of (2.15) exist and we get
\begin{align}
&\left(\nabla \varphi_k u, \nabla \varphi_k u\right) + \left(q^+ \varphi_k u / \varphi_k k u\right) - \left(q^- \varphi_k u / \varphi_k k u\right) + \left(\varphi_k u, \varphi_k u\right) = \\
&\left((\nabla \varphi_k)^2 u, \varphi_k u\right) + \left((u \nabla \varphi_k - \varphi_k \nabla u, \varphi_k \varphi_k \varphi_k)\right). \quad (2.16)
\end{align}

Since, from Step 2, we know \( \varphi_k u \in Q(T_F) \), we conclude from (2.16) and from \( T_F + 1 \geq 1 \) that
\[ \left\| \varphi_k u \right\|^2 \leq \left( (T_F + 1) \varphi_k u / \varphi_k u \right) = \text{RHS of (2.16)} \rightarrow 0 \quad (k \rightarrow \infty). \]

Thus \( u = 0 \), which proves Step 5.

Since \( T_F \) is self-adjoint by Step 5, the theorem is proven.

COROLLARY 1. Let \( k \in \mathbb{N} \). Assume (C_1) and (C_2). Set \( q^+_k := q^+ \);
\[ q^-_k(x) := \begin{cases} 
  q^-(x) & \text{if } |x| \leq k \\
  0 & \text{if } |x| > k
\end{cases} \]
\[ q^-_k := q^+_k - q^-_k; \]
and define \( T_k \) and \( T_{\text{max}} \) as in (1.6) and (1.2). Assume additionally
\[ T_k \text{ is self-adjoint} \quad \text{(C_4)} \]
and
\[ \text{there exist } 0 \leq a_k < 1 \text{ and } b_k \geq 0 \text{ such that} \quad \text{(C_5)} \]
\[ |(q^-_k w / w)| \leq a_k (-\Delta, w) + b_k |w|^2, \quad w \in C^\infty_0(\mathbb{R}^n). \quad (2.17) \]

Then \( T_{\text{max}} \) is self-adjoint.

PROOF. (C_3) holds trivially. From (2.17), we deduce
\[ (-\Delta, w) + (q^-_k w / w) \leq \frac{1}{1 - a_k} \left\{ (T_k w / w) + (b_k + 1) |w|^2 \right\} \]
which implies (2.1). Since \( C^\infty_0(\mathbb{R}^n) \) is dense in \( H^1(\mathbb{R}^n) \cap Q(q^+) \) in the sense of the
norm \| \cdot \|_{t_k^+} \) (as we know from [5], see Step 2 above), (2.17) implies that \( C^\infty_0(\mathbb{R}^n) \) is a form core of \( t_k \). Therefore, \( (C_5) \) holds and, by the theorem, self-adjointness of \( t_{\max} \) follows.

Note that, for \( q \in L^2_{\text{loc}}(\mathbb{R}^n) \), Corollary 1 implies the result of Simader [1] since then \( t_{\min}^* = t_{\max} \) where

\[
T_{\max} := T_{\max} \big| C^\infty_0(\mathbb{R}^n).
\]

COROLLARY 2. Let \( k \in \mathbb{N} \). Assume \((C_1)\) and \((C_2)\). Set

\[
q_k(x) := \begin{cases} 
q(x) & \text{if } |x| \leq k \\
0 & \text{if } |x| > k
\end{cases}
\]

and define \( t_k \) and \( t_{\max} \) as in (1.6) and (1.2). Assume additionally \((C_4)\) and \((C_5)\). Then \( t_{\max} \) is self-adjoint. The proof follows immediately from the theorem.

In the case \( q \in L^2_{\text{loc}}(\mathbb{R}^n) \), Corollary 2 implies the result of Brézis [2] by the same arguments as above. We also should note that, if \( q_k^+ = q^+ \) and \( q_k^- = q^- \) \((k \in \mathbb{N})\) and if \( q^- \) is form-bounded relative to the form of \((-\Delta + q^+)\) with bound \( < 1 \), our theorem is Kato's [3] result for the semibounded case. In fact, our proof is a variant of Kato's proof of his main theorem in [3].

Note: On leave from: Technische Universität Berlin, Fachbereich Mathematik Straße des 17 Juni 135, 1 Berlin 12, Germany

REFERENCES


5. SIMON, B. Maximal and minimal Schrödinger operators and forms, J. Operator Theory appl. 1 (1979), 37-47.
