RESEARCH NOTES

INFORMATION SETS AS PERMUTATION CYCLES FOR QUADRATIC RESIDUE CODES

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ABSTRACT. The two cases $p = 7$ and $p = 23$ are the only known cases where the automorphism group of the $[p + 1, (p + 1)/2]$ extended binary quadratic residue code, $O(p)$, properly contains $PSL(2, p)$. These codes have some of their information sets represented as permutation cycles from $\text{Aut}(Q(p))$. Analysis proves that all information sets of $Q(7)$ are so represented but those of $Q(23)$ are not.

KEY WORDS AND PHRASES. Binary code, Automorphism group of a code, Information set, Mathieu group of degree 24, Orbit.

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1. INTRODUCTION.

In an earlier article [1] it was noted that information sets of a binary extended quadratic residue code often appear as cycles of permutations from the automorphism group of the code. The present study of two important cases ($p = 7$ and $p = 23$) decides whether, for each code, every information set can be found as a cycle of a code preserving permutation. These results have immediate application in the area of decoding since several techniques of decoding depend for success on knowledge of a code's information sets. Our work suggests a technique, faster than an exhaustive search, for obtaining all information sets for the codes in question.

Let p be a prime with $p \equiv \pm 1 \pmod{8}$. If Q and N signify the non zero quadratic residues and non-residues mod p , define polynomials in $GF(2)[X]$ by

$$\begin{aligned} E_Q(x) &= \sum_{r \in Q} x^r \\ E_N(x) &= \sum_{n \in N} x^n \end{aligned} \quad \text{if } p \equiv -1 \pmod{8}$$

and

$$\begin{aligned} E_Q(x) &= 1 + \sum_{r \in Q} x^r \\ E_N(x) &= 1 + \sum_{n \in N} x^n \end{aligned} \quad \text{if } p \equiv 1 \pmod{8}$$

where E_Q and E_N are idempotents for codes of dimension $(p+1)/2$ and length p over the binary field. These codes, when extended by adding an overall parity check, become the $[p+1, (p+1)/2]$ extended quadratic residue codes $Q(p)$ and $N(p)$ hereafter referred to as QR codes. See [2] - [6].

The coordinate places of QR codes have a standard label set $L = \{0, 1, 2, \dots, p-1, \infty\}$. A subset M , of L , is called strictly dependent for $Q(p)$ whenever the columns of a generator matrix for $Q(p)$ indicated by M sum to the zero column. Recall that $c \in Q(p)^\perp$ iff the coordinates on which c is non zero form a strictly dependent subset of L . Remember too that $PSL(2, p) \leq \text{Aut}(Q(p))$ for all p under consideration and define any permutation on L from $\text{Aut}(Q(p))$ as a $(p+1)/2$ element if it has two cycles each of length $(p+1)/2$.

$Q(7)$ is a self dual, doubly even (i.e., all codewords have weight $\equiv 0 \pmod{4}$), $[8, 4]$ code with automorphism group of order 1344. There are fourteen codewords of weight four and one each of weights zero and eight.

Of the $\binom{8}{4} = 70$ four element subsets (4-sets) of $L = \{0, 1, \dots, 6, \infty\}$, the fourteen codewords of weight four specify the strictly dependent 4-sets of $Q(7)$ ($=Q(7)^\perp$) so that 56 information sets remain. At most 42 of these information sets could be found as cycles of 4-elements from $PSL(2, 7)$. Indeed each 4-element of $PSL(2, 7)$ belongs to one of the 21 different conjugate cyclic subgroups (see [7]) of $PSL(2, 7)$. Each such subgroup splits L into two orbits (4-sets) which are either both information sets or both strictly dependent (see [1]). Even if each subgroup gave a unique splitting and each half of a splitting were an information set, at most 42 information sets would be accounted for.

Acting on all 70 4-sets with $\text{PSL}(2,7)$ causes three orbits of 4-sets to arise. The CAMAC system indicates the fourteen strictly dependent 4-sets form one orbit and the 42 information sets which are cycles of 4-elements of $\text{PSL}(2,p)$ form a second orbit. The third orbit consists of the 14 information sets remaining. Acting on all 70 4-sets with the full automorphism group of $Q(7)$, all 56 information sets appear in one orbit, proving that each information set of $Q(7)$ is expressed as a cycle of a 4-element from $\text{Aut}(Q(7))$.

$Q(23)$ is the $[24, 12]$ code known as the extended Golay code. Its full automorphism is the Mathieu group $M(24)$ of order $(24)(23)(22)(21)(20)(48)$. In spite of the size of $\text{Aut}(Q(23))$ we shall prove that some of the information sets of $Q(23)$ are not expressed as cycles of 12-elements from $\text{Aut}(Q(23)) (=M(24))$.

The self dual code $Q(23)$ has 759 codewords of weight 8 (called octads) and the same number of weight 16 codewords. Besides the single zero codevector and the all ones codeword, there are 2576 weight 12 codewords known as dodecads. Conway [8] supplies much of the background for the following argument.

The action of $M(24)$ on the 12-sets of L partitions the collection of 12-sets into the 5 orbits named below:

- $U(12)$ - the 12-sets which are each a dodecad
- $S(12)$ - the 12-sets which each contain exactly 1 octad
- $S(12+)$ - the 12-sets which each contain exactly 3 octads
- $U(12-)$ - the 12-sets each of which differs from a dodecad in exactly one coordinate
- $T(12)$ - the 12-sets in none of the above orbits.

An information set of $Q(23)$, say K , and its complement $L-K$ both belong to the same orbit since $p \equiv -1 \pmod{8}$. See [1]. Furthermore, $M(24)$'s character table ([9]) implies all 12-elements of $M(24)$ are conjugate in $M(24)$. The conclusion is that all information sets of $Q(23)$ which are cycles of 12-elements from $M(24)$ must belong to the same orbit.

The orbits $S(12)$, $S(12+)$, and $U(12)$ consist of 12-sets which contain strictly dependent sets of $Q(23)$. The weight distribution of $Q(23)$ implies that these are the only ways a codeword from $Q(23)$ may be obtained in a 12-set. So the

remaining two orbits, $T(12)$ and $U(12-)$, are seen to hold information sets of $Q(23)$. And it becomes clear from the earlier conclusion that only one of these two orbits holds the $Q(23)$ information sets which appear as cycles of 12-elements from $M(24)$. The natural question is "Which orbit is this?"

The following argument shows why the information sets of $U(12-)$ are not cycles of 12-elements from $M(24)$. Suppose K is a 12-set of $U(12-)$ and $u \in Q(23)$ is a dodecad having 11 ones on K and a single one on $L-K$. For the sake of contradiction, now suppose that there is some 12-element, say g , in $M(24)$ having K and $L-K$ as its two cycles. If that were the case, the vector $u + g(u)$ would be a codeword of $Q(23)$ of weight four--a clear impossibility. So our result is proved.

Thus $T(12)$ must hold the information sets of $Q(23)$ appearing as cycles of 12-elements from $M(24)$.

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