UNIVALENCE OF NORMALIZED SOLUTIONS OF $W''(z) + p(z)W(z) = 0$

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(Received August 27, 1981)

ABSTRACT. Denote solutions of $W''(z) + p(z)W(z) = 0$ by $W_\alpha(z) = z^\alpha[1 + \sum_{n=1}^{\infty} a_n z^n]$ and $W_\beta(z) = z^\beta[1 + \sum_{n=1}^{\infty} b_n z^n]$, where $0 < \Re\{\beta\} \leq 1/2 \leq \Re\{\alpha\}$ and $z^2 p(z)$ is holomorphic in $|z| < 1$. We determine sufficient conditions on $p(z)$ so that $[W_\alpha(z)]^{1/\alpha}$ and $[W_\beta(z)]^{1/\beta}$ are univalent in $|z| < 1$.

KEY WORDS AND PHRASES. Univalent, spirallike, starlike.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. Primary - 34A20; Secondary - 30C45.

1. INTRODUCTION.

Consider the differential equation

$$W''(z) + p(z)W(z) = 0, \text{ where} \quad (1.1)$$

$$z^2 p(z) = p_0 + p_1 z + \cdots + p_n z^n + \cdots, \quad p_0 \neq 0, \quad (1.2)$$

is holomorphic for $|z| < 1$.

The indicial equation associated with the regular singular point of the equation (1.1) at the origin is

$$\lambda^2 - \lambda + p_0 = 0 \quad (1.3)$$

and has roots which we designate by α and β , where $\alpha + \beta = 1$ and $\Re(\alpha) \geq 1/2 \geq \Re(\beta)$.

We will also use the notation

$$\alpha = \alpha_1 + i\alpha_2, \beta = \beta_1 + i\beta_2. \quad (1.4)$$

Corresponding to the root α there is always a unique solution of (1.1) of the form

$$W_\alpha(z) = z^\alpha [1 + \sum_{n=1}^{\infty} a_n z^n] \quad (1.5)$$

valid for $|z| < 1$.

We restrict our attention in this paper to those β for which $\beta_1 = \Re(\beta) > 0$. We then obtain a unique solution of (1.1) of the form

$$W_\beta(z) = z^\beta [1 + \sum_{n=1}^{\infty} b_n z^n] \quad (1.6)$$

valid for $|z| < 1$.

We define two normalizations $F_\alpha(z)$ and $F_\beta(z)$ of the solutions of (1.5) and (1.6) as follows:

$$\begin{aligned} F_\alpha(z) &= [W_\alpha(z)]^{1/\alpha} = z + \dots, \\ F_\beta(z) &= [W_\beta(z)]^{1/\beta} = z + \dots \end{aligned} \quad (1.7)$$

where we choose that branch of each function for which the derivative at the origin is 1.

Next we consider the "comparison" equation

$$W''(z) + p_C^*(z)W_C(z) = 0, \text{ with} \quad (1.8)$$

$$z^2 p_C^*(z) \equiv C(z^2 p^*(z) - p_0^*) + p_0^*, \quad C > 0, \quad (1.9)$$

$$\text{and where } z^2 p^*(z) = p_0^* + p_1^* z + \dots + p_n^* z^n + \dots$$

is non-constant and holomorphic for $|z| < 1$ with p_i^* , $i = 0, 1, 2, \dots$ real and $p_0^* \leq 1/4$. With these restrictions on $z^2 p^*(z)$ the solutions of (1.8) are real on the real axis (see [1]). We will designate the exponents associated with the regular singular point of (1.8) at the origin by α^* and β^* , where $\alpha^* + \beta^* = 1$ and $\alpha^* \geq 1/2 \geq \beta^*$. As in the case of equation (1.1) we obtain for any α^* a unique solution of (1.8) of the form

$$W_{\alpha^*, C}^*(z) = z^{\alpha^*} \left[1 + \sum_{n=1}^{\infty} a_n^*(C) z^n \right] \quad (1.10)$$

valid for $|z| < 1$, and for any $\beta^* > 0$ a unique solution of the form

$$W_{\beta^*, C}^* = z^{\beta^*} \left[1 + \sum_{n=1}^{\infty} b_n^*(C) z^n \right] \quad (1.11)$$

valid for $|z| < 1$.

In [1] Robertson determined fairly general sufficient conditions on $p(z)$ relative to $p^*(z)$ under which $F_{\alpha}(z)$ is univalent in $|z| < 1$. In [2] Brown extended these results to $F_{\beta}(z)$ but only for real β satisfying $0 < \beta \leq 1/2$. In the Main Theorem of this paper we present sharp sufficient conditions on $p(z)$ relative to $p_C^*(z)$ under which the function $F_{\beta}(z)$ is univalent and spirallike in $|z| < 1$, where β may be complex valued. We then compare these results to those of Robertson for $F_{\alpha}(z)$.

2. PRELIMINARIES.

S will denote the class of functions $f(z)$ holomorphic and univalent in the unit disk $D \equiv \{z: |z| < 1\}$ and normalized so that $f(0) = 0$, $f'(0) = 1$.

We shall say that $f(z) \in F_{\varphi, \alpha}$ if and only if for some real number φ , $|\varphi| < \pi/2$, and some α , $0 \leq \alpha < 1$,

$$\Re \left\{ \frac{e^{i\varphi} z f'(z)}{f(z)} \right\} > \alpha$$

for all $z \in D$. $F_{\varphi} \equiv F_{\varphi, 0}$ is the class of functions called spirallike in D , [3], [4]. Functions in the subclass $S^*(\alpha) \equiv F_{0, \alpha}$ are called starlike of order α in D . $S^*(0)$ is the class of functions starlike in D . It follows that $S^*(\alpha) \subset F_{\varphi} \subset S$; (see [2]).

We will need the following result.

THEOREM 2.1. Let $z^2 p_C^*(z)$, $W_{\alpha^*, C}^*(z)$, and $W_{\beta^*, C}^*(z)$ be defined by (1.9), (1.10), and (1.11) respectively. If for all $|z| < 1$

$$\Re \{ z^2 p^*(z) \} \leq |z|^2 p^*(|z|)$$

then for fixed C $\frac{|z| W_{\alpha^*, C}^*(|z|)}{W_{\alpha^*, C}^*(|z|)}$ is monotonic decreasing for all $|z| < \min(1, R_{\alpha^*}^*(C))$,

and $\frac{|z|W'_{\beta^*,C}(|z|)}{W_{\beta^*,C}(|z|)}$ is monotonic decreasing for all $|z| < \min(1, R_{\beta^*,C})$ where $R_{\alpha^*,C}$ and $R_{\beta^*,C}$ are the smallest positive zeros of the functions $W'_{\alpha^*,C}(r)$ and $W'_{\beta^*,C}(r)$ respectively.

In the case of α^* this result is given on page 262 of [1]. For β^* the result follows from (3.16) and Theorem 3.18 of [2] after noting that if $z^2 p^*(z)$ is non-constant the equality $\Re\{z^2 p^*(z)\} = |z|^2 p^*(|z|)$ cannot hold for all $0 \leq r \leq r_1$ on any ray $\theta = \text{constant} \neq 0$.

The condition that $z^2 p^*(z)$ be nonconstant is necessary to ensure strict monotonicity in the results above since if $z^2 p^*(z)$ is constant so are

$$\frac{|z|W'_{\alpha^*,C}(|z|)}{W_{\alpha^*,C}(|z|)} \quad \text{and} \quad \frac{|z|W'_{\beta^*,C}(|z|)}{W_{\beta^*,C}(|z|)}.$$

3. LEMMAS.

In this section we prove the lemmas used to obtain Theorem A and The Main Theorem in section 4.

Since all of the results of this section are stated for $W_{\beta}(z)$ and $W_{\beta^*,C}(z)$ we adopt the following notational convention:

$$\begin{aligned} W &\equiv W(z) \equiv W_{\beta}(z) = z^{\beta} \left[1 + \sum_{n=1}^{\infty} b_n z^n \right], \\ W_C &\equiv W_C(z) \equiv W_{\beta^*,C}(z) = z^{\beta^*} \left[1 + \sum_{n=1}^{\infty} b_n^*(C) z^n \right]. \end{aligned} \quad (3.1)$$

It is important to note that all of the results of this section remain valid if W and W_C are replaced by either $W_{\alpha}(z)$ and $W_{\alpha^*,C}(z)$ or by $W_{\alpha}(z)$ and $W_{\beta^*,C}(z)$ and, moreover, the proofs are obtained by making corresponding changes in the proofs given here.

In our lemmas we will investigate the rate of change of $\Re\{\frac{zW'}{W}\}$ and $\Im\{\frac{zW'}{W}\}$ on rays issuing from the origin. For this reason we designate z by $re^{i\theta}$, fix θ , vary r , and use (1.1) to obtain

$$r \frac{d}{dr} \left[\frac{zW'}{W} \right] = -z^2 p(z) + \frac{zW'}{W} - \left[\frac{zW'}{W} \right]^2, \quad (3.2)$$

where W' designates differentiation with respect to z .

Taking real and imaginary parts of (3.2) we obtain

$$r \frac{d}{dr} \Re \left\{ \frac{zW'}{W} \right\} = -\Re \{ z^2 p(z) \} + \Re \left\{ \frac{zW'}{W} \right\} - \Re^2 \left\{ \frac{zW'}{W} \right\} + \Im^2 \left\{ \frac{zW'}{W} \right\} \quad (3.3)$$

and

$$r \frac{d}{dr} \Im \left\{ \frac{zW'}{W} \right\} = -\Im \{ z^2 p(z) \} + \Im \left\{ \frac{zW'}{W} \right\} - 2\Re \left\{ \frac{zW'}{W} \right\} \Im \left\{ \frac{zW'}{W} \right\}. \quad (3.4)$$

Also from (1.8) and the fact that W_C is real for real z , we obtain for $z \geq 0$

$$r \frac{d}{dr} \left(\frac{rW'_C(r)}{W_C(r)} \right) = -r^2 p_C^*(r) + \frac{rW'_C(r)}{W_C(r)} - \left(\frac{rW'_C(r)}{W_C(r)} \right)^2. \quad (3.5)$$

Our goal is to determine conditions on $z^2 p(z)$ relative to $z^2 p_C^*(z)$ which will ensure that on every ray $\theta = \text{constant}$

$$\Re \left\{ \frac{zW'}{W} \right\} - \frac{rW'_C(r)}{W_C(r)} \geq 0 \text{ for all } 0 \leq r < 1. \quad (3.6)$$

Then it will follow from (1.7) that for $\theta = \text{constant}$

$$\Re \left\{ \frac{\beta z F'_\beta(z)}{F_\beta(z)} \right\} - \frac{rW'_C(r)}{W_C(r)} \geq 0 \text{ for all } 0 \leq r < 1. \quad (3.7)$$

Since $\Re\{\beta\} > 0$ (3.7) implies $F_\beta(z)$ is univalent and spirallike in $|z| < R(C)$, where $R(C)$ is the smallest positive zero of $W'_C(r)$ or 1 whichever is the smaller. We will show that C can be adjusted so that $R(C) = 1$ and, therefore, $F_\beta(z)$ is univalent in D .

In [1] the inequality (3.6) was obtained when $W = W_\alpha(z)$ and $W_C(r) = W_{\alpha^*, C}(r)$ by a method that relied upon the inequality

$$r \frac{d}{dr} \Re \left\{ \frac{zW'}{W} \right\} \geq -\Re \{ z^2 p(z) \} + \Re \left\{ \frac{zW'}{W} \right\} - \Re^2 \left\{ \frac{zW'}{W} \right\}$$

obtained from (3.3) by neglecting the term $\Im^2 \left\{ \frac{zW'}{W} \right\}$. Unfortunately this inequality is not sharp enough to yield (3.6) when $W = W_\beta(z)$ and $W_C(r) = W_{\beta^*, C}(r)$ by the method of [1]. In this paper we retain the term $\Im^2 \left\{ \frac{zW'}{W} \right\}$ in (3.3) and derive estimates for

its rate of growth relative to that of $\frac{rW'_C}{W_C}$. These estimates enable us to establish (3.6) for $W = W_{\beta}(z)$ and $W_C(r) = W_{\beta, C}^*(r)$.

We introduce the following notation where $z = re^{i\theta}$, θ is constant, and r satisfies the inequalities $0 \leq r < 1$.

$$T(r) \equiv \Re\left\{\frac{zW'_C}{W}\right\} - r \frac{W'_C(r)}{W_C(r)}. \quad (3.8)$$

$$S(r) \equiv -\Re\left\{\frac{zW'_C}{W}\right\} - r \frac{W'_C(r)}{W_C(r)}. \quad (3.9)$$

$$M(r) \equiv -\Im\left\{\frac{zW'_C}{W}\right\} - r \frac{W'_C(r)}{W_C(r)}. \quad (3.10)$$

$$N(r) \equiv \Im\left\{\frac{zW'_C}{W}\right\} - r \frac{W'_C(r)}{W_C(r)}. \quad (3.11)$$

$$\tau(r) \equiv -\Re\{z^2 p(z)\} + r^2 p_C^*(r). \quad (3.12)$$

$$\sigma(r) \equiv \Re\{z^2 p(z)\} + r^2 p_C^*(r). \quad (3.13)$$

$$\mu(r) \equiv \Im\{z^2 p(z)\} + r^2 p_C^*(r). \quad (3.14)$$

$$\nu(r) \equiv -\Im\{z^2 p(z)\} + r^2 p_C^*(r). \quad (3.15)$$

$$R(C) \text{ is the smallest positive zero of } W'_C(r) \quad (3.16)$$

$$R^* = \min(1, R(C)). \quad (3.17)$$

In terms of this notation our goal is to establish conditions under which $T(r) > T(0)$ on every ray $\theta = \text{constant}$, $|z| < R^*$.

From (3.3), (3.4), and (3.5) we obtain the following relations:

$$r \frac{dT(r)}{dr} = \tau(r) + T(r)\left(1 - \frac{2r W'_C(r)}{W_C(r)}\right) - T^2(r) + \Im^2\left\{\frac{zW'_C}{W}\right\}. \quad (3.18)$$

$$r \frac{dS(r)}{dr} = \sigma(r) + S(r)\left(1 + \frac{2r W'_C(r)}{W_C(r)}\right) + S^2(r) - \Im^2\left\{\frac{zW'_C}{W}\right\} + 2\left(\frac{r W'_C(r)}{W_C(r)}\right)^2. \quad (3.19)$$

$$r \frac{dM(r)}{dr} = \mu(r) + M(r) + 2 R \left(\frac{zW'}{W} \right) \mathfrak{I} \left(\frac{zW'}{W} \right) + \left(\frac{rW'_C(r)}{W_C(r)} \right)^2 \quad (3.20)$$

$$r \frac{dN(r)}{dr} = \nu(r) + N(r) - 2 R \left(\frac{zW'}{W} \right) \mathfrak{I} \left(\frac{zW'}{W} \right) + \left(\frac{rW'_C(r)}{W_C(r)} \right)^2. \quad (3.21)$$

The proofs of most of the lemmas in this section reflect a common simple theme that is set forth formally in the following lemma.

LEMMA A. Let $G(r)$ be a real-valued differentiable function on $a \leq r \leq b$. Let $G(r) > 0$ for all $a \leq r < \rho \leq b$ and $G(\rho) = 0$. Then it follows that $G'(\rho) \leq 0$.

It should be noted that from their definitions it follows that the functions $T(r)$, $S(r)$, $M(r)$ and $N(r)$ can assume a value at most a finite number of times on any segment $\theta = \text{constant}$, $0 \leq r \leq r_2 < 1$. Thus if, for example, $T(r) = k$ for some r in $0 \leq r \leq r_2 < 1$, then there is a smallest $r \equiv \rho$ in this interval for which $T(\rho) = k$.

LEMMA 1. Let $z^{2p^*}(z)$ satisfy the conditions of Theorem 2.1. If for fixed θ we have

- a) $\tau(r) \geq \tau(0)$ for all $0 < r < 1$,
- b) $T(r) > T(0)$ for all $0 \leq r \leq r_1 < R^*$,
- c) $\frac{r_1 W'_C(r_1)}{W_C(r_1)} = \beta^* - |\beta_2|$,
- d) $|\beta_2| \leq 2(\beta_1 - \beta^*)$,

then $T(r) > T(0)$ for all $0 < r < R^*$.

PROOF. Assume that the conclusion is false. Then there exists an r , $r_1 < r < R^*$, for which $T(r) - T(0) = 0$. Let ρ be the smallest such r . Then since $T(r) - T(0) > 0$ for all $0 < r < \rho$ it follows from Lemma A, with $G(r) = T(r) - T(0)$, that $\left. \frac{dT(r)}{dr} \right|_{r=\rho} \leq 0$. We will show, however, that our hypotheses imply that $\left. \frac{dT(r)}{dr} \right|_{r=\rho} > 0$. Thus there can be no roots of $T(r) - T(0)$ on $r_1 < r < R^*$, and consequently $T(r) > T(0)$ for all $0 < r < R^*$.

From (3.18) and a) and b) of our hypotheses we have

$$r \left. \frac{dT(r)}{dr} \right|_{r=\rho} > \tau(0) + T(0) \left(1 - \frac{2\rho W'_C(\rho)}{W_C(\rho)} \right) - T^2(0). \quad (3.22)$$

From Theorem 2.1 it follows that $f(r) \equiv 1 - \frac{2rW'_C(r)}{W_C(r)}$ is monotonic increasing on $r_1 < r < R^*$. Thus

$$\begin{aligned} r \frac{dT(r)}{dr} \Big|_{r=p} &> \tau(0) + T(0)f(r_1) - T^2(0) \\ &= \tau(0) + T(0)(1 - 2\beta^*) + 2|\beta_2|(\beta_1 - \beta^*) - T^2(0), \end{aligned}$$

which by d) of our hypotheses is

$$\begin{aligned} &\geq \tau(0) + T(0)(1 - 2\beta^*) + \beta_2^2 - T^2(0) \\ &= r \frac{dT(r)}{dr} \Big|_{r=0} = 0. \end{aligned}$$

Thus $\frac{dT(r)}{dr} \Big|_{r=p} > 0$. This is the desired contradiction from which it follows that $T(r) > T(0)$ for all $0 < r < R^*$.

LEMMA 2. Let $z^*_p(z)$ satisfy the conditions of Theorem 2.1. Let $\beta_1 - \beta^* > 0$, and let r_1 satisfy the inequalities $0 < r_1 < R^*$. If for fixed θ we have $S(r_1) > S(0)$, and for all $0 < r_1 \leq r \leq r_2 < R^*$

$$a) \sigma(r) \geq \sigma(0),$$

$$b) \mathcal{J}\left\{\frac{zW'}{W}\right\} \leq \beta_2^2$$

then $S(r) > S(0)$ for all $r_1 \leq r \leq r_2$.

PROOF. From (3.19) and a) and b) of our hypotheses we have

$$r \frac{dS(r)}{dr} \Big|_{r=p} \geq \sigma(0) + S(0) + S^2(0) + \frac{2\rho W'_C(\rho)}{W_C(\rho)} S(0) + 2\left(\frac{\rho W'_C(\rho)}{W_C(\rho)}\right)^2 - \beta_2^2. \quad (3.23)$$

Now use the method of proof of Lemma 1 with $G(r) \equiv S(r) - S(0)$, ρ the smallest zero of $G(r)$ on $r_1 \leq r \leq r_2$, and

$$f(r) \equiv \frac{2rW'_C(r)}{W_C(r)} S(0) + 2\left(\frac{rW'_C(r)}{W_C(r)}\right)^2.$$

From Theorem 2.1 it follows that if $\beta_1 - \beta^* > 0$ then $f(r)$ is monotonic increasing on $0 < r \leq r_2$. Thus from (3.22) we have

$$r \frac{dS(r)}{dr} \Big|_{r=p} > \sigma(0) + S(0) + S^2(0) + f(0) - \beta_2^2 = r \frac{dS(r)}{dr} \Big|_{r=0} = 0,$$

and the lemma follows as in the proof of Lemma 1.

LEMMA 3A. Let $z_p^*(z)$ satisfy the conditions of Theorem 2.1. Let $0 < \beta_2 \leq \beta_1 - \beta^*$, and let r_1 satisfy the inequalities $0 < r_1 < R^*$. If for a fixed θ $S(r_1) > S(0)$, $N(r_1) > N(0)$, and if for all $0 < r_1 \leq r \leq r_2 < R^*$ we have

$$a) \sigma(r) \geq \sigma(0),$$

$$b) v(r) \geq v(0),$$

$$c) 0 \leq \Re\left\{\frac{zW'}{W}\right\} \leq \beta_2,$$

then it follows that $N(r) > N(0)$ for all $r_1 \leq r \leq r_2$.

PROOF. From Lemma 2 it follows that $S(r) > S(0)$ for all $r_1 \leq r \leq r_2$, and from (3.9) we have

$$\Re\left\{\frac{zW'}{W}\right\} < -S(0) - \frac{rW'_C(r)}{W_C(r)} = \beta_1 + \beta^* - \frac{rW'_C(r)}{W_C(r)} \text{ for all } r_1 \leq r \leq r_2.$$

Using this inequality along with (3.21) and c) of our hypotheses we have

$$r \frac{dN(r)}{dr} > v(r) + N(r) - 2(\beta_1 + \beta_2 - \frac{rW'_C(r)}{W_C(r)})(N(r) + \frac{rW'_C(r)}{W_C(r)}) + (\frac{rW'_C(r)}{W_C(r)})^2 \quad (3.24)$$

where we have used the definition (3.11) in the third term.

From (3.24) we obtain

$$r \frac{dN(r)}{dr} > v(r) + N(r)[1 - 2(\beta_1 + \beta^*)] + 2N(r) \frac{rW'_C(r)}{W_C(r)} - 2(\beta_1 + \beta^*) \frac{rW'_C(r)}{W_C(r)} + 3(\frac{rW'_C(r)}{W_C(r)})^2. \quad (3.25)$$

Now use the method of proof of Lemma 1 with $G(r) \equiv N(r) - N(0)$, ρ the smallest zero of $G(r)$ in $r_1 \leq r \leq r_2$, and

$$f(r) \equiv 2[N(0) - (\beta_1 + \beta^*)] \frac{rW'_C(r)}{W_C(r)} + 3(\frac{rW'_C(r)}{W_C(r)})^2.$$

Then from (3.25), Theorem 2.1, and a) and b) of our hypotheses we have

$$r \frac{dN(r)}{dr} \Big|_{r=\rho} > v(0) + N(0)[1 - 2(\beta_1 + \beta^*)] + f(\rho). \quad (3.26)$$

From Theorem 2.1 it follows that if $\beta_2 \leq \beta_1 - \beta^*$ then $f(r)$ is monotonic increasing on $0 \leq r \leq r_2$. Thus from (3.26) we have

$$\begin{aligned} r \frac{dN(r)}{dr} \Big|_{r=0} &> v(0) + N(0)[1 - 2(\beta_1 + \beta^*)] + f(0) \\ &= v(0) + N(0) - 2\beta_1\beta_2 + \beta^{*2} \\ &= r \frac{dN(r)}{dr} \Big|_{r=0} = 0. \end{aligned}$$

The lemma now follows as in the proof of Lemma 1.

COROLLARY 3A. Let $z_p^*(z)$ satisfy the conditions of Theorem 2.1. Let $0 < \beta_2 \leq \beta_1 - \beta^*$, and let r_1 satisfy the inequalities $0 < r_1 < R^*$. If for a fixed θ

$$\mathcal{J}\left\{\frac{r_1 e^{i\theta} W'(r_1 e^{i\theta})}{W(r_1 e^{i\theta})}\right\} > 0, \quad S(r_1) > S(0), \quad N(r_1) > N(0), \quad \text{and if for all}$$

$0 < r_1 \leq r \leq r_2 < R^*$ we have

$$a) \quad \sigma(r) \geq \sigma(0),$$

$$b) \quad v(r) \geq v(0),$$

$$c) \quad \mathcal{J}\left\{\frac{zW'}{W}\right\} \leq \beta_2,$$

$$d) \quad \frac{rW'_C(r)}{W_C(r)} \geq \max(\beta^* - \beta_2, 0),$$

then it follows that $\mathcal{J}\left\{\frac{zW'}{W}\right\} > 0$ and $N(r) > N(0)$ for all $r_1 \leq r \leq r_2$.

PROOF. To prove that $\mathcal{J}\left\{\frac{zW'}{W}\right\} > 0$ for all $r_1 \leq r \leq r_2$ note that if ρ is the smallest zero of $\mathcal{J}\left\{\frac{zW'}{W}\right\}$ in the interval $r_1 < r < r_2$, then we can apply Lemma 3A on the interval $r_1 \leq r \leq \rho$ to obtain $N(r) > N(0)$ for all $r_1 \leq r \leq \rho$. Then from (3.11) it follows that

$$\mathcal{J}\left\{\frac{zW'}{W}\right\} > \beta_2 - \beta^* + \frac{rW'_C(r)}{W_C(r)} \quad (3.27)$$

for all $r_1 \leq r \leq \rho$. (3.27) and d) of our hypotheses give

$$\beta_2 - \beta^* + \frac{rW'_C(r)}{W_C(r)} \geq 0 \quad \text{on } r_1 \leq r \leq \rho. \quad (3.28)$$

Then (3.27) and (3.28) imply that

$$\Re \left\{ \frac{\rho e^{i\theta} W'(\rho e^{i\theta})}{W(\rho e^{i\theta})} \right\} > 0$$

which contradicts the assumption on ρ . Thus $\Re \left\{ \frac{zW'}{W} \right\} \geq 0$ for all $r_1 \leq r \leq r_2$. Now from Lemma 3A it follows directly that $N(r) > N(0)$ for all $r_1 \leq r \leq r_2$.

LEMMA 3B. Let $z^2 p^*(z)$ satisfy the conditions of Theorem 2.1. Let $\beta_2 < 0$, $|\beta_2| \leq \beta_1 - \beta^*$, and let r_1 satisfy the inequalities $0 < r_1 < R^*$. If for a fixed θ we have $M(r_1) > M(0)$, $S(r_1) > S(0)$, and if for all $0 < r_1 \leq r \leq r_2 < R^*$ we have

- a) $\sigma(r) \geq \sigma(0)$,
- b) $\mu(r) \geq \mu(0)$,
- c) $\beta_2 \leq \Re \left\{ \frac{zW'}{W} \right\} \leq 0$,

then it follows that $M(r) > M(0)$ for all $r_1 \leq r \leq r_2$.

PROOF. The method of proof is the same as that of Lemma 3A. Start with (3.20) instead of (3.21) and replace the condition $0 < \beta_2 \leq \beta_1 - \beta^*$ by $|\beta_2| \leq \beta_1 - \beta^*$. The lemma then follows by establishing the contradiction $\frac{dM}{dr} \Big|_{r=0} > 0$.

COROLLARY 3B. Let $z^2 p^*(z)$ satisfy the conditions of Theorem 2.1. Let $\beta_2 < 0$, $|\beta_2| \leq \beta_1 - \beta^*$, and let r_1 satisfy the inequalities $0 < r_1 < R^*$. If for a fixed θ ,

$$\Re \left\{ \frac{r_1 e^{i\theta} W'(r_1 e^{i\theta})}{W(r_1 e^{i\theta})} \right\} < 0, \quad S(r_1) > S(0), \quad M(r_1) > M(0), \quad \text{and if for all}$$

$0 < r_1 \leq r \leq r_2 < R^*$ we have

- a) $\sigma(r) \geq \sigma(0)$,
- b) $\mu(r) \geq \mu(0)$,
- c) $\Re \left\{ \frac{zW'}{W} \right\} \geq \beta_2$,
- d) $\frac{rW'_C(r)}{W_C(r)} \geq \max(\beta^* - |\beta_2|, 0)$,

then it follows that $M(r) > M(0)$ for all $r_1 \leq r \leq r_2$.

PROOF. The method of proof is the same as that of Corollary 3A using $M(r)$ in place of $N(r)$ throughout.

LEMMA 4A. Let $z^2 p^*(z)$ satisfy the conditions of Theorem 2.1. Let

$0 < \beta_2 \leq \frac{\beta_1 - \beta^*}{2}$, and let r_1 satisfy the inequalities $0 < r_1 < R^*$. If for a fixed θ $M(r_1) > M(0)$, and if for all $0 < r_1 \leq r \leq r_2 < R^*$ we have

$$a) \mu(r) \geq \mu(0),$$

$$b) T(r) \geq T(0),$$

$$c) \mathfrak{J}\left\{\frac{rW'}{W}\right\} \geq \beta_2,$$

$$d) \frac{rW'_C(r)}{W_C(r)} \geq \max(\beta^* - \beta_2, 0),$$

then it follows that $M(r) > M(0)$ for all $r_1 \leq r \leq r_2$.

PROOF. From (3.20), (3.8), and a) and b) of our hypotheses it follows that

$$r \frac{dM(r)}{dr} \geq \mu(0) + M(r) + 2(T(0) + \frac{rW'_C(r)}{W_C(r)} \mathfrak{J}\left\{\frac{rW'}{W}\right\} + \left(\frac{rW'_C(r)}{W_C(r)}\right)^2$$

with equality for $r = 0$. Using definition (3.10) we rewrite this inequality in the form

$$r \frac{dM(r)}{dr} \geq \mu(0) + M(r)(1 - 2T(0)) - 2(M(r) + T(0)) \frac{rW'_C(r)}{W_C(r)} - \left(\frac{rW'_C(r)}{W_C(r)}\right)^2 \quad (3.29)$$

Now use the method of proof of Lemma 1 with $G(r) = M(r) - M(0)$, ρ the smallest zero of $G(r)$ on $r_1 \leq r \leq r_2$, and

$$f(r) \equiv -2(M(0) + T(0)) \frac{rW'_C(r)}{W_C(r)} - \left(\frac{rW'_C(r)}{W_C(r)}\right)^2.$$

From Theorem 2.1 it follows that if $\beta_2 \leq (\beta_1 - \beta^*)/2$ then $f(r)$ is monotonic increasing on $0 < r \leq r_2$. Then from (3.29) we have

$$\begin{aligned} r \frac{dM(r)}{dr} \Big|_{r=\rho} &\geq \mu(0) + M(0)(1 - 2T(0)) - 2(M(0) + T(0)) \left(\frac{\rho W'_C(\rho)}{W_C(\rho)}\right) - \left(\frac{\rho W'_C(\rho)}{W_C(\rho)}\right)^2 \\ &> \mu(0) + M(0)(1 - 2T(0)) - 2(M(0) + T(0))\beta^* - \beta^{*2} \\ &= r \frac{dM(r)}{dr} \Big|_{r=0} = 0, \end{aligned}$$

and the lemma now follows as in the proof of Lemma 1.

LEMMA 4B. Let $z_p^*(z)$ satisfy the conditions of Theorem 2.1. Let $\beta_2 < 0$, $|\beta_2| \leq \frac{\beta_1 - \beta^*}{2}$, and let r_1 satisfy the inequalities $0 < r_1 < R^*$. If for a fixed θ $N(r_1) > N(0)$, and if for all $0 < r_1 \leq r \leq r_2 < R^*$ we have

- a) $v(r) \geq v(0)$,
- b) $T(r) \geq T(0)$,
- c) $\mathcal{J}\{\frac{zW'}{W}\} \leq \beta_2$
- d) $\frac{rW'_C(r)}{W_C(r)} \geq \max(\beta^* - |\beta_2|, 0)$,

then it follows that $N(r) > N(0)$ for all $r_1 \leq r \leq r_2$.

PROOF. The proof proceeds precisely as that of Lemma 4A except that (3.11) is used in place of (3.10), and the condition $|\beta_2| \leq \frac{\beta_1 - \beta^*}{2}$ replaces $\beta_2 \leq \frac{\beta_1 - \beta^*}{2}$.

LEMMA 5A. Let $z_p^*(z)$ satisfy the conditions of Theorem 2.1. Let $0 < \beta_2 \leq \frac{\beta_1 - \beta^*}{2}$, and let r_1 satisfy the inequalities $0 < r_1 < R^*$. If for a fixed θ $S(r_1) > S(0)$, $M(r_1) > M(0)$, and if for all $0 < r_1 \leq r \leq r_2 < R^*$ we have

- a) $\mu(r) \geq \mu(0)$,
- b) $\sigma(r) \geq \sigma(0)$,
- c) $T(r) \geq T(0)$,
- d) $\mathcal{J}\{\frac{zW'}{W}\} \geq \beta_2$,
- e) $\frac{rW'_C(r)}{W_C(r)} \geq \max(\beta^* - \beta_2, 0)$,

then it follows that $S(r) > S(0)$ for all $r_1 \leq r \leq r_2$.

PROOF. By Lemma 4A we have

$$\mathcal{J}\{\frac{zW'}{W}\} \leq -M(0) - \frac{rW'_C(r)}{W_C(r)} = \beta_2 + \beta^* - \frac{rW'_C(r)}{W_C(r)} > 0 \quad (3.30)$$

for all $r_1 \leq r \leq r_2$. Thus from (3.19) and (3.30) it follows that

$$r \frac{dS(r)}{dr} \geq \sigma(r) + S(r) \left(1 + \frac{2rW'_C(r)}{W_C(r)}\right) + S^2(r) - (-M(0) - \frac{rW'_C(r)}{W_C(r)})^2 + 2\left(\frac{rW'_C(r)}{W_C(r)}\right)^2. \quad (3.31)$$

Now use the method of proof of Lemma 1 with $G(r) \equiv S(r) - S(0)$, ρ the smallest zero of $G(r)$ on $r_1 \leq r \leq r_2$ and

$$f(r) \equiv 2(S(0) - M(0))\left(\frac{rW'_C(r)}{W_C(r)}\right) + \left(\frac{rW'_C(r)}{W_C(r)}\right)^2.$$

From Theorem 2.1 it follows that if $\beta_2 \leq (\beta_1 - \beta^*)/2$ then $f(r)$ is monotonic increasing on $0 < r \leq r_2$. Then from (3.31) and a) through d) of our hypotheses it follows that

$$\begin{aligned} \left. \frac{rdS(r)}{dr} \right|_{r=0} &\geq \sigma(0) + S(0) + S^2(0) - M^2(0) + f(0) \\ &= \left. \frac{rdS(r)}{dr} \right|_{r=0} = 0, \end{aligned}$$

and our lemma follows as in the proof of Lemma 1.

LEMMA 5B. Let $z^2 p^*(z)$ satisfy the conditions of Theorem 2.1. Let $\beta_2 < 0$, $|\beta_2| \leq (\beta_1 - \beta^*)/2$, and let r_1 satisfy the inequalities $0 < r_1 < R^*$. If for a fixed θ , $N(r_1) > N(0)$, $S(r_1) > S(0)$, and if for all $0 < r_1 \leq r \leq r_2 < R^*$ we have

- a) $v(r) \geq v(0)$,
- b) $\sigma(r) \geq \sigma(0)$,
- c) $T(r) \geq T(0)$,
- d) $\mathcal{J}\left\{\frac{zW'}{W}\right\} \leq \beta_2$,
- e) $\frac{rW'_C(r)}{W_C(r)} \geq \max(\beta^* - |\beta_2|, 0)$,

then it follows that $S(r) > S(0)$ for all $r_1 \leq r \leq r_2$.

PROOF. The proof proceeds precisely as that of Lemma 5A except that Lemma 4B is used in place of Lemma 4A, and the condition $|\beta_2| \leq (\beta_1 - \beta^*)/2$ replaces $\beta_2 \leq (\beta_1 - \beta^*)/2$.

LEMMA 6. Let $z^2 p^*(z)$ satisfy the conditions of Theorem 2.1. Let r_1 satisfy the inequalities $0 < r_1 < R^*$. If for fixed θ $T(r_1) > T(0)$, and if for all $0 < r_1 \leq r \leq r_2 < R^*$ we have

- a) $\tau(r) \geq \tau(0)$,
- b) $\mathcal{J}^2\left\{\frac{zW'}{W}\right\} \geq \beta_2^2$,

then $T(r) > T(0)$ for all $r_1 \leq r \leq r_2$.

PROOF. With $G(r)$ and ρ defined as in Lemma 1, we obtain from (3.18) and our hypotheses

$$\begin{aligned} \left. \frac{rdT(r)}{dr} \right|_{r=\rho} &> \tau(0) + T(0)(1 - 2\beta^*) - T^2(0) + \beta_2^2 \\ &= \left. \frac{rdT(r)}{dr} \right|_{r=\rho} = 0, \end{aligned}$$

and the lemma follows as in the proof of Lemma 1.

LEMMA 7A. Let $z^2 p^*(z)$ satisfy the conditions of Theorem 2.1. Let $0 < \beta_2 \leq \beta_1 - \beta^*$, and let r_1 satisfy the inequalities $0 < r_1 < R^*$. If for a fixed θ $S(r_1) > S(0)$, $N(r_1) > N(0)$, $T(r_1) > T(0)$, $\Re \left\{ \frac{r_1 e^{i\theta} W'(r_1 e^{i\theta})}{W(r_1 e^{i\theta})} \right\} > 0$; and if for all $0 < r_1 \leq r \leq r_2 < R^*$ we have

- a) $\sigma(r) \geq \sigma(0)$,
- b) $\tau(r) \geq \tau(0)$,
- c) $v(r) \geq v(0)$,
- d) $\Re \left\{ \frac{zW'}{W} \right\} \leq \beta_2$
- e) $\frac{rW'_C(r)}{W_C(r)} \geq \max(\beta^* - \beta_2, 0)$,

then $T(r) > T(0)$ for all $r_1 \leq r \leq r_2$.

PROOF. From (3.11) and e) of our hypotheses it follows that

$(N(0) + \frac{rW'_C(r)}{W_C(r)}) \geq 0$ for all $r_1 \leq r \leq r_2$. Then from (3.18) and Corollary 3A we have

$$\frac{rdT}{dr} \geq \tau(r) + T(r)(1 - 2r \frac{W'_C(r)}{W_C(r)}) - T^2(r) + (N(0) + \frac{rW'_C(r)}{W_C(r)})^2 \quad (3.32)$$

for all $r_1 \leq r \leq r_2$.

Now use the method of proof of Lemma 1 with $G(r)$ and ρ defined as in Lemma 1 and with

$$f(r) \equiv -2(T(0) - N(0)) \frac{rW'_C(r)}{W_C(r)} + \left(\frac{rW'_C(r)}{W_C(r)} \right)^2.$$

From Theorem 2.1 it follows that if $\beta_2 \leq \beta_1 - \beta^*$ then $f(r)$ is monotonic increasing on $0 < r \leq r_2$. Thus, from (3.32) and b) of our hypotheses we have

$$\begin{aligned}
\left. \frac{rdT(r)}{dr} \right|_{r=0} &> \tau(0) + T(0) - T^2(0) + N^2(0) + f(0) \\
&= \tau(0) + T(0)(1 - 2\beta^*) - T^2(0) + (N(0) + \beta^*)^2 \\
&= \left. \frac{rdT(r)}{dr} \right|_{r=0} = 0,
\end{aligned}$$

and our lemma follows as in the proof of Lemma 1.

LEMMA 7B. Let $z^2 p^*(z)$ satisfy the condition of Theorem 2.1. Let r_1 satisfy the inequalities $0 < r_1 < R^*$ and let $\beta_2 < 0$, $|\beta_2| \leq \beta_1 - \beta^*$. If for a fixed θ $S(r_1) > S(0)$, $M(r_1) > M(0)$, $T(r_1) > T(0)$, $\mathcal{J}\left\{\frac{r_1 e^{i\theta} w'_\beta(r_1 e^{i\theta})}{w_\beta(r_1 e^{i\theta})}\right\} < 0$, and if for all

$0 < r_1 \leq r \leq r_2 < R^*$ we have

- a) $\sigma(r) \geq \sigma(0)$,
- b) $\tau(r) \geq \tau(0)$,
- c) $\mu(r) \geq \mu(\theta)$,
- d) $\mathcal{J}\left\{\frac{zW'}{W}\right\} \geq \beta_2$,
- e) $\frac{rW'_C(r)}{W_C(r)} \geq \max(\beta^* - |\beta_2|,)$,

then $T(r) > T(0)$ for all $r_1 \leq r \leq r_2$.

PROOF. The proof proceeds precisely as that of Lemma 7A except that Corollary 3B is used in place of Corollary 3A.

LEMMA 8. If for all θ , $0 \leq \theta \leq 2\pi$, and for all $0 < r < R^*$ we have

a) $\sigma(r) \geq \sigma(0)$, b) $\tau(r) \geq \tau(0)$, c) $\mu(r) \geq \mu(0)$, d) $\nu(r) \geq \nu(0)$ then it follows that

$$e) \left. \frac{dS}{dr} \right|_{r=0} \geq 0, \quad f) \left. \frac{dT}{dr} \right|_{r=0} \geq 0, \quad g) \left. \frac{dM}{dr} \right|_{r=0} \geq 0 \quad \text{and} \quad h) \left. \frac{dN}{dr} \right|_{r=0} \geq 0$$

where $a \Rightarrow e$, $b \Rightarrow f$, $c \Rightarrow g$, and $d \Rightarrow h$. Moreover, strict inequality holds in e) through h) if $\beta_1 > \beta^* > 0$.

PROOF. We prove that $b \Rightarrow f$. All four implications can be proved using the same techniques.

For $0 \leq r < R^*$ we have

$$\frac{zW'}{W} = \beta + c_1 z + \dots + c_n z^n + \dots \text{ and} \quad (3.33)$$

$$\frac{zW'_C(z)}{W_C(z)} = \beta^* + c_1^* z + \dots + c_n^* z^n + \dots \quad (3.34)$$

where $c_1 = \frac{-p_1}{2\beta}$ and $c_1^* = \frac{-Cp_1^*}{2\beta^*}$ with p_1 and p_1^* defined by (1.2) and (1.9)

respectively.

Now $\tau(r) \geq \tau(0)$ if and only if $-\Re\{z^2 p(z) - p_0\} + r^2 p_C^*(r) - p_0^* \geq 0$. Therefore, if $\tau(r) \geq \tau(0)$ for all θ , $0 \leq \theta \leq 2\pi$, and all $0 \leq r < R^*$, it follows that $(-\Re\{p_1 z\} + Crp_1^*) \geq 0$ for sufficiently small r and for all θ .

Thus we must have

$$-\Re\{p_1 e^{i\theta}\} + Cp_1^* \geq 0 \text{ for all } \theta. \quad (3.35)$$

Now from (3.33) and (3.34) we have

$$\begin{aligned} \frac{dT}{dr}\bigg|_{r=0} &= \Re\{c_1 e^{i\theta}\} - c_1^* \\ &= \Re\left\{\frac{-p_1 e^{i\theta}}{2\beta}\right\} + \frac{Cp_1^*}{2\beta^*}. \end{aligned}$$

Then if we write β in the form $|\beta|e^{i\varphi}$, we have

$$\frac{dT}{dr}\bigg|_{r=0} = \Re\left\{\frac{-p_1 e^{i(\theta - \varphi)}}{2|\beta|}\right\} + \frac{Cp_1^*}{2\beta^*}. \quad (3.36)$$

$$\text{Thus } \frac{dT}{dr}\bigg|_{r=0} \geq 0 \text{ if and only if } -\Re\{p_1 e^{i(\theta - \varphi)}\} + \frac{Cp_1^*}{\left(\frac{\beta}{|\beta|}\right)^*} \geq 0 \quad (3.37)$$

for all θ .

However, since $\beta_1 \geq \beta^* > 0$, we have $0 < \beta^*/|\beta| \leq 1$ and it follows from (3.35) and (3.37) that $\frac{dT}{dr}\bigg|_{r=0} \geq 0$ with strict inequality if $\beta_1 > \beta^*$.

4. THEOREM A AND THE MAIN THEOREM.

We have designated the first result of this section as Theorem A since it is our analog of Theorem A of [1] when $\gamma = 0$.

THEOREM A. Let $W \equiv W(z) \equiv W_{\beta}(z) = z^{\beta} [1 + \sum_{n=1}^{\infty} b_n z^n]$ be the unique solution of

$W''(z) + p(z)W(z) = 0$ where

$$z^2 p(z) = p_0 + p_1 z + \dots + p_n z^n + \dots$$

is holomorphic in $|z| < 1$, and $\beta = \beta_1 + i\beta_2$ with $0 < \beta_1 \leq 1/2$.

Let $W_C \equiv W_C(z) \equiv W_{\beta, C}^*(z) = z^{\beta^*} [1 + \sum_{n=1}^{\infty} b_n^*(C) z^n]$ be the unique solution of

$W''(z) + z^2 p_C^*(z)W(z) = 0$ where

$$z^2 p_C^*(z) = C[z^2 p^*(z) - p_0^*] + p_0^* \text{ with}$$

$$z^2 p^*(z) = p_0^* + p_1^* z + \dots + p_n^* z^n + \dots, \quad p_0^* \leq 1/4,$$

holomorphic in $|z| < 1$ and real on the real axis; and where $C > 0$ and $0 < \beta^* \leq 1/2$.

Let $R(C)$ be the smallest positive root of $W_C'(r)$, $0 < r < 1$, if such exists.

For $|z| < 1$ let $z^2 p(z)$ and $z^2 p^*(z)$ satisfy the inequalities

$$(i) \quad \Re\{z^2 p^*(z)\} \leq |z|^2 p^*(|z|),$$

$$(ii) \quad |z^2 p(z) - p_0| \leq |z|^2 p_C^*(|z|) - p_0^*.$$

Then for $|\beta_2| \leq (\beta_1 - \beta^*)/2$ it follows that

$$\Re\left\{\frac{zW'}{W}\right\} \geq \frac{rW_C'(r)}{W_C(r)} > 0$$

for all $|z| = r < R^* \equiv \min(R(C), 1)$.

We first note that (i) of our hypotheses ensures that $z^2 p^*(z)$ satisfies the conditions of Theorem 2.1. In addition (ii) implies inequalities a) through c) of Lemma 8. For example, inequality a) of Lemma 8 is valid if and only if

$$\Re\{z^2 p(z)\} + |z|^2 p_C^*(|z|) \geq \Re\{p_0\} + p_0^*,$$

and this is true if and only if

$$\Re\{z^2 p(z) - p_0\} \geq -\{|z|^2 p_C^*(|z|) - p_0^*\}$$

which in turn follows from (ii).

We will obtain the result of Theorem A by proving that $T(r) > T(0)$ on any ray $\theta = \text{constant}$, $0 < r < R^*$. To establish this fact we will need Lemmas 1 through 7 whose hypotheses require us to know whether $\mathcal{J}\{\frac{zW'}{W}\} \geq \beta_2$ or $\mathcal{J}\{\frac{zW'}{W}\} \leq \beta_2$. Therefore, we introduce on each ray $\theta = \text{constant}$, $0 < r < R^*$, the points ρ_i where ρ_i and ρ_{i+1} , $\rho_i < \rho_{i+1}$, are consecutive values of r at which $\mathcal{J}\{\frac{zW'}{W}\} - \beta_2$ changes sign. We then show that $T(r) > T(0)$ on every interval $\rho_i \leq r \leq \rho_{i+1}$.

The proof requires consideration of the following four cases.

Case 1: $0 < \beta_2 < \frac{\beta_1 - \beta^*}{2}$, θ constant, $\frac{d}{dr}[\mathcal{J}\{\frac{zW'}{W}\}]_{r=0} > 0$,

Case 2: $0 < \beta_2 < \frac{\beta_1 - \beta^*}{2}$, θ constant, $\frac{d}{dr}[\mathcal{J}\{\frac{zW'}{W}\}]_{r=0} < 0$,

Case 3: $\beta_2 < 0$, $|\beta_2| < \frac{\beta_1 - \beta^*}{2}$, θ constant, $\frac{d}{dr}[\mathcal{J}\{\frac{zW'}{W}\}]_{r=0} > 0$,

Case 4: $\beta_2 < 0$, $|\beta_2| < \frac{\beta_1 - \beta^*}{2}$, θ constant, $\frac{d}{dr}[\mathcal{J}\{\frac{zW'}{W}\}]_{r=0} < 0$.

For $\beta_2 = 0$ the result of Theorem A was established by Robertson [1] for $W = W_\alpha(z)$ and $W_C = W_{\alpha^*, C}(z)$ and by Brown [2] for $W = W_\beta(z)$, $W_C = W_{\beta^*, C}(z)$.

We will assume that $\frac{rW'_C(r)}{W_C(r)} > \beta^* - |\beta_2|$ for all $0 \leq r < R^*$ since if for some ρ , $0 < \rho < R^*$, we have $\frac{\rho W'_C(\rho)}{W_C(\rho)} = \beta^* - |\beta_2|$, we can restrict our attention to the interval $0 \leq r < \rho$ and then use Lemma 1 on the interval $\rho < r < R^*$.

PROOF OF CASE 1.

1. From Lemma 8 it follows that there exists a ρ^* , $0 < \rho^* < \rho_1$, such that for all θ $S(r) > S(0)$, $T(r) > T(0)$, $M(r) > M(0)$ and $N(r) > N(0)$ for all $0 < r \leq \rho^*$.
2. Now fix θ and apply Lemma 6 to the interval $\rho^* \leq r \leq \rho$ to obtain $T(r) > T(0)$ on $\rho^* \leq r \leq \rho_1$.
3. From definitions (3.10) and (3.11), the definition of the ρ_i and the monotonicity of $\frac{rW'_C(r)}{W_C(r)}$ on $0 < r < R^*$, it follows that $M(\rho_1) > M(0)$ and $N(\rho_1) > N(0)$.

4. Then by Lemma 5A applied to the interval $\rho^* \leq r \leq \rho_1$ we have $S(\rho_1) > S(0)$.
5. By Lemma 7A it then follows that $T(r) > T(0)$ on the interval $\rho_1 \leq r \leq \rho_2$.
6. By Lemma 6, $T(r) > T(0)$ on the interval $\rho_2 \leq r \leq \rho_3$.
7. By 4 above and Lemma 2 we have $S(\rho_2) > S(0)$.
8. As in 3 above we obtain $M(\rho_2) > M(0)$, $N(\rho_2) > N(0)$ and $N(\rho_3) > N(0)$.
9. Then by Lemma 5A we have $S(\rho_3) > S(0)$.
10. From 8, 9 and Lemma 7A it follows that $T(r) > T(0)$ on the interval $\rho_3 \leq r \leq \rho_4$.

By successive iterations of steps 6 through 10 it follows that if

$T(r) > T(0)$ on $\rho_i \leq r \leq \rho_{i+1}$ then $T(r) > T(0)$ on $\rho_{i+1} \leq r \leq \rho_{i+2}$. Moreover, the proof actually demonstrates that $T(r) > T(0)$ on any interval of the ray $\theta = \text{constant}$, $0 < r < R^*$, on which either $\mathcal{J}\{\frac{ZW'}{W}\} - \beta_2 \geq 0$ or $\mathcal{J}\{\frac{ZW'}{W}\} - \beta_2 \leq 0$. Thus it follows that $T(r) > T(0) = \beta_1 - \beta^* \geq 0$ for all $0 < r < R^*$ on any ray $\theta = \text{constant}$.

PROOF OF CASE 2.

1. As in step 1 of Case 1 we have that there exists a ρ^* , $0 < \rho^* < \rho_1$, such that for all θ , $S(r) > S(0)$, $T(r) > T(0)$, $M(r) > M(0)$ and $N(r) > N(0)$ for all $0 \leq r \leq \rho^*$.
2. By Lemma 7A it then follows that $T(r) > T(0)$ on the interval $\rho^* \leq r \leq \rho_1$.
3. By Lemma 6 we have $T(r) > T(0)$ on $\rho_1 \leq r \leq \rho_2$.
4. By definition (3.10) and the monotonicity of $\frac{rW'_C(r)}{W_C(r)}$ on $0 < r < R^*$ we have $M(\rho_1) > M(0)$.
5. By 1 above and Lemma 2 we have $S(\rho_1) > S(0)$.
6. Then by Lemma 5A it follows that $S(\rho_2) > S(0)$.
7. By definition (3.11) and the monotonicity of $\frac{rW'_C(r)}{W_C(r)}$ on $0 < r < R^*$ we have $N(\rho_2) > N(0)$.
8. Then by 3, 6, 7, and Lemma 7A we have $T(r) > T(0)$ on the interval $\rho_2 \leq r \leq \rho_3$.

By successive iteration of steps 3 through 8 (omitting the reference to step 1 in step 5) it follows that if $T(r) > T(0)$ on $\rho_i \leq r \leq \rho_{i+1}$ then $T(r) > T(0)$ on $\rho_{i+1} \leq r \leq \rho_{i+2}$. Then as in Case 1 we obtain $T(r) > T(0) = \beta_1 - \beta^* \geq 0$ for all

$0 < r < R^*$ on any ray $\theta = \text{constant}$.

PROOFS OF CASE 3 AND CASE 4. The proofs of Case 3 and Case 4 are identical to those of Case 2 and Case 1 respectively except that all A lemmas are replaced by corresponding B lemmas.

COROLLARY A. Theorem A remains true if $W(z)$ and $W_C(z)$ are replaced by either $W_\alpha(z)$ and $W_{\alpha^*,C}^*(z)$ or by $W_\beta(z)$ and $W_{\beta^*,C}^*(z)$.

PROOF. The result follows from the fact that all of the lemmas used in the proof of Theorem A remain valid under the indicated substitutions.

Our Main Theorem will be derived from Theorem A precisely as the Main Theorem of [1] was derived from Theorem A of [1]. We will not reproduce Robertson's proofs but simply mention that his methods apply equally well to $W_C(z) = W_{\alpha^*,C}^*(z)$ and $W_C(z) = W_{\beta^*,C}^*(z)$, and then summarize the needed results in the following lemma.

LEMMA 4.1. Let $z^2 p^*(z)$ satisfy the conditions of Theorem 2.1 and let R be fixed, $0 < R < 1$. Then there exists a $C \equiv C(R) > 0$ such that when $p_C^*(z) \equiv p_{C(R)}^*(z)$ we have $W'_{C(R)}(R) = 0$ and $W'_{C(R)}(r) > 0$ for all $0 < r < R$. Moreover, for fixed $z^2 p^*(z)$ we have $\lim_{R \rightarrow 1} C(R) \equiv A(p^*) \equiv A$ is finite and $W'_A(r) > 0$ for all $0 < r < 1$. The value A , called the universal constant corresponding to $z^2 p^*(z)$, is largest in the sense that for any $\epsilon > 0$ there exists an $r(\epsilon)$, $0 < r(\epsilon) < 1$, such that $W'_A(r(\epsilon)) = 0$ and $W'_{A+\epsilon}(r(\epsilon)) \leq 0$.

THE MAIN THEOREM. Let

$$z^2 p^*(z) = p_0^* + p_1^* z + \dots + p_n^* z^n + \dots$$

be nonconstant and holomorphic in $|z| < 1$ and real on the real axis with $p_0^* \leq 1/4$.

Let

$$\Re\{z^2 p^*(z)\} \leq |z|^2 p^*(|z|) \text{ for } |z| < 1. \quad (4.1)$$

Let $z^2 p_A^*(z) = A(z^2 p^*(z) - p_0^*) + p_0^*$ where $A = A(p^*)$ is the universal constant corresponding to $z^2 p^*(z)$. Let

$$W_A(z) \equiv W_{\beta^*,A}^*(z) = z^{\beta^*} \left[1 + \sum_{n=1}^{\infty} b_n^*(C) \right], \quad |z| < 1, \quad \beta^* > 0,$$

be the unique solution of

$$W''(z) + p_A^*(z)W(z) = 0$$

corresponding to the smaller root of the indicial equation. Then the function

$$F_A(z) \equiv [W_A(z)]^{1/\beta^*} = z + \dots$$

is a holomorphic function, univalent and starlike in $|z| < 1$, and is not both holomorphic and univalent in any larger circle whenever $A > 0$.

Let $z^2 p(z)$ be holomorphic in $|z| < 1$ with

$$|z^2 p(z) - p_0| \leq |z|^2 p_A^*(|z|) - p_0^* \quad (4.2)$$

for all $|z| < 1$. Let

$$W(z) \equiv W_\beta(z) = z^\beta [1 + \sum_{n=1}^{\infty} b_n z^n], \quad |z| < 1,$$

$\beta = \beta_1 + i\beta_2$, $\beta_1 > 0$, be the unique solution of

$$W''(z) + p(z)W(z) = 0$$

corresponding to the root β , with smaller real part, of the indicial equation.

Then if $|\beta_2| \leq \frac{\beta_1 - \beta^*}{2}$ the function

$$F_\beta(z) = [W_\beta(z)]^{1/\beta} = z + \dots$$

is a holomorphic function, univalent and spirallike in $|z| < 1$. The constant

$A = A(p^*)$ is the largest possible one.

PROOF. From Theorem A we have

$$\Re\left\{\frac{\beta z F'_\beta(z)}{F_\beta(z)}\right\} = \Re\left\{\frac{z W'_\beta(z)}{W_\beta(z)}\right\} \geq \frac{|z| W'_A(|z|)}{W_A(|z|)} > 0, \quad |z| < 1. \quad (4.3)$$

Now if we choose $z^2 p(z) \equiv z^2 p_C^*(z)$ then $\beta_2 = 0$, $W_\beta(z) \equiv W_{\beta^*, C}(z) \equiv W_C(z)$ and from Theorem 3.23 of [2] we have

$$\Re\left\{\frac{z W'_C(z)}{W_C(z)}\right\} \geq \frac{|z| W'_C(|z|)}{W_C(|z|)}, \quad |z| < R(C). \quad (4.4)$$

Thus from (4.4) and the definition of A we have

$$\Re\left\{\frac{zF'_A(z)}{F_A(z)}\right\} \geq \frac{1}{\beta^*} \frac{|z|W'_A(|z|)}{W_A(|z|)} > 0, \quad |z| < 1. \quad (4.5)$$

From (4.3) it follows that $F_\beta(z)$ is univalent and spirallike in $|z| < 1$, and from (4.5) it follows that $F_A(z)$ is univalent and starlike in $|z| < 1$. Moreover, since equality holds in (4.4) when z is real and positive, and since $W'_{A+\epsilon}(R) = 0$ for some R , $0 < R < 1$, for arbitrarily small positive values of ϵ , it is clear that $F_{A+\epsilon}(z)$ is not univalent in $|z| < 1$ no matter how small a positive ϵ we take. Thus the constant A is the largest possible. The proof that the radius of univalence of $F_A(z)$ is precisely 1 is contained in [1] page 265 and will not be reproduced here.

COROLLARY B. The Main Theorem is true when $W = W_\alpha(z)$ and $W_C(z) = W_{\alpha^*, C}(z)$ whenever $|\alpha_2| \leq \frac{\alpha_1 - \alpha^*}{2}$ and also when $W = W_\alpha(z)$ and $W_C(z) = W_{\beta^*, C}(z)$ whenever $|\alpha_2| \leq \frac{\alpha_1 - \beta^*}{2}$.

The proof of this corollary is immediate since all of our previous results remain valid under the indicated substitutions for $W(z)$ and $W_C(z)$.

5. REMARKS.

We will now indicate how the results of our Main Theorem and Corollary B compare to or extend those of Robertson's Main Theorem in [1] for $\gamma = 0$.

In [1] the condition

$$\Re\{z^2 p(z)\} \leq |z|^2 p_A^*(|z|) \quad (4.6)$$

replaces our condition (4.2), there is no explicit bound on $|\alpha_2|$, and the results refer to the case $W = W_\alpha(z)$ and $W_C = W_{\alpha^*, C}(z)$.

Our condition (4.2) is independent of condition (4.6). When $\alpha^* \geq 1/2$ our Corollary B requires that

$$|\alpha_2| \leq \frac{\alpha_1}{2} - \frac{1}{4} \quad (4.7)$$

while (4.6) implies that

$$\alpha_1 - \alpha_1^2 + \alpha_2^2 \leq \alpha^* - \alpha^{*2}$$

so that

$$|\alpha_2| \leq \alpha_1 - \frac{1}{2}. \quad (4.8)$$

We refer now to the "exponent plane" of Figure 1 in which α_1 and β_1 are measured horizontally and α_2 and β_2 are measured vertically. Our conclusions are the following:

1. In region I only our Main Theorem applies.
2. In regions II only Robertson's Main Theorem applies.
3. In regions III $\cup$ IV Robertson's Main Theorem applies and our Corollary B applies with $W = W_{\alpha}(z)$ and $W_C = W_{\beta^*, C}^*(z)$.
4. In region IV Robertson's Main Theorem applies and our Corollary B applies with $W = W_{\alpha}(z)$ and $W_C = W_{\alpha^*, C}^*(z)$.

Thus it is in region I that we have an extension of the results of [1] and [2].

Figure 1

We note that we can obtain Robertson's Theorem A with $\gamma = 0$ from (3.18) with $W = W_{\alpha}(z)$ and $W_C = W_{\alpha^*, C}^*(z)$. This follows by noting first that the hypotheses of Theorem A of [1] imply that $\alpha_1 \geq \alpha^*$ and that $\tau(r) \geq 0$ for all $0 \leq r < 1$. Also, if $\alpha_1 > \alpha^*$ then $T(0) > 0$, while if $\alpha_1 = \alpha^*$ then $\alpha_2 = 0$ and as in the proof of Lemma 8 it follows that $T'(0) > 0$. Now if we let ρ be the smallest zero of $T(r)$ on $0 < r < 1$ we obtain from (3.18)

$$\frac{\rho dT(\rho)}{dr} = \tau(\rho) + \mathcal{J}^2 \left\{ \frac{\rho e^{i\theta} W'_\beta(\rho e^{i\theta})}{W_\beta(\rho e^{i\theta})} \right\}. \quad (4.9)$$

Thus either $\frac{dT(\rho)}{d\rho} > 0$ or both terms in the right member of (4.9) vanish. The former conclusion yields an immediate contradiction to the definition of ρ . If we assume the latter conclusion then an examination of the successive derivatives of (3.18) shows that the first non-vanishing derivative of $T(r)$ at ρ is positive and of even order. Thus $T(r) \geq 0$ for all $0 < r < R^*$.

Finally we point out that for $\gamma \neq 0$, $|\gamma| < \pi/2$, Robertson's Theorem A can be obtained by applying the same reasoning as above to the following analog of (3.18) where $W = W_\alpha(z)$ and $W_C = W_{\alpha, C}^*(z)$.

$$\frac{rdT}{dr} = \tau(r) + T(r) \left(1 - \frac{2rW'_C(r)}{W_C(r)} \right) - \sec \gamma T^2(r) + \sec \gamma \mathcal{J}^2 \left(\frac{zW'}{W} \right).$$

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