RESEARCH NOTES

ALMOST CONVEX METRICS AND
PEANO COMPACTIFICATIONS

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(Received June 23, 1981)

ABSTRACT. Let \((X,d)\) denote a locally connected, connected separable metric space. We say the space is S-metrizable provided there is a topologically equivalent metric \(\rho\) on \(X\) such that \((X,\rho)\) has Property S, i.e., for any \(\varepsilon > 0\), \(X\) is the union of finitely many connected sets of \(\rho\)-diameter less than \(\varepsilon\). It is well-known that S-metrizable spaces are locally connected and that if \(\rho\) is a Property S metric for \(X\), then the usual metric completion \((\tilde{X},\tilde{\rho})\) of \((X,\rho)\) is a compact, locally connected, connected metric space; i.e., \((\tilde{X},\tilde{\rho})\) is a Peano compactification of \((X,\rho)\). In an earlier paper, the author conjectured that if a space \((X,d)\) has a Peano compactification, then it must be S-metrizable. In this paper, that conjecture is shown to be false; however, the connected spaces which have Peano compactifications are shown to be exactly those having a totally bounded, almost convex metric. Several related results are given.

KEY WORDS AND PHRASES. Almost Convex Metrics, Property S metrics, Peano spaces, Compactifications.

1980 MATHEMATICS SUBJECT CLASSIFICATIONS CODES. 54F25, 54E35.

1. INTRODUCTION.

Throughout this note let \((X,d)\) denote a metric space. We say that \(d\) is convex
provided that, for any pair \( x, y \in X \), there is \( z \in X \) such that \( d(x, z) = d(z, y) = d(x, y)/2 \).

It is almost convex if, for \( x, y \in X \) and \( \varepsilon > 0 \), there is \( z \in X \) such that \( |d(x, z) - d(x, y)/2| < \varepsilon \) and \( |d(z, y) - d(x, y)/2| < \varepsilon \) \([1,2]\).

We say that \( X \) is \( S \)-metrizable provided there is a topologically equivalent metric \( \rho \) on \( X \) such that \((X, \rho)\) has Property \( S \), i.e., for any \( \varepsilon > 0 \), \( X \) is the union of finitely many connected sets of \( \rho \)-diameter less than \( \varepsilon \). It is well-known that \( S \)-metrizable spaces are locally connected and that if \( \rho \) is a Property \( S \) metric for \( X \), then the usual metric completion \((\tilde{X}, \tilde{\rho})\) of \((X, \rho)\) is a compact, locally connected, connected metric space, i.e., \((\tilde{X}, \tilde{\rho})\) is a Peano compactification of \((X, \rho)\) \([3, \text{p. 154}]\).

It is a famous result of R. H. Bing that any continuous curve \( P \) (i.e., a compact, locally connected, connected metric space) can be assigned a convex metric \([1]\).

In an earlier paper \([4]\), the author conjectured that, if \( X \) is locally connected and if \( X \) has a Peano compactification, then \( X \) is \( S \)-metrizable. In this paper we show, by example, that this conjecture is false; however, we do obtain a characterization of such spaces in terms of the existence of a totally bounded, almost convex metric for \( X \). We also obtain several related results characterizing totally bounded (\( S \)-metrizable, almost convex) metrics.

2. PEANO COMPACTIFICATIONS.

THEOREM 2.1. A connected metric space \((X, d)\) has a Peano compactification if and only if it has a topologically equivalent totally bounded, almost convex metric.

PROOF. The necessity. Let \((P, r)\) be a Peano compactification of \( X \), i.e., \( P \) is a continuous curve and \( X \) is a dense subset of \( P \). By R. H. Bing's result, there exists an equivalent metric \( \rho \) for \( P \) such that \( \rho \) is convex. It then follows that \( \sigma = r | X \) is totally bounded and almost convex; cf. \([1, \text{Thm. 10}]\).

The Sufficiency. Let \( r \) be an almost convex, totally bounded metric for \( X \). Let \((\tilde{X}, \tilde{r})\) be the usual metric completion of \((X, r)\). We will argue that \((\tilde{X}, \tilde{r})\) is a Peano compactification of \((X, r)\). Clearly, \( \tilde{X} \) is compact since \( r \) is totally bounded.

Furthermore, \( \tilde{r} \) is a convex metric for \( \tilde{X} \); let \( x, y \in \tilde{X} \). Since \( r \) is almost convex, there exists a sequences \( x_1, x_2, \ldots, y_1, y_2, \ldots \), and \( z_1, z_2, \ldots \) in \( X \) such that

\[
|r(x_n, z_n) - r(x_n, y_n)/2| < 2^{-n} \quad \text{and} \quad |r(z_n, y_n) - r(x_n, y_n)/2| < 2^{-n}.
\]
Since \( r \) is totally bounded, without loss of generality, we may assume that each of the sequences \( x_1, x_2, ..., y_1, y_2, ..., \) and \( z_1, z_2, ... \) is Cauchy in \( X \). Then by the completeness of \((X, r)\), it follows that \( \lim_{n \to \infty} x_n = x \) and \( \lim_{n \to \infty} y_n = y \). Furthermore, if \( \lim_{n \to \infty} z_n = z \), then \( r(x, z) = r(z, y) = r(x, y)/2 \) since \( r \) is continuous. Thus \( r \) is convex and complete. It follows from Theorem 3.1 of [5] that the spheres \( S_r(x, \varepsilon) \) of \( X \) are connected sets. This implies that \( X \) is locally connected and this completes the proof.

**EXAMPLE 2.1.** Let \( P \) be the square \( \{ (x, y) \in \mathbb{R}^2 : 0 \leq x, y \leq 1 \} \) in the plane. For \( n \in \mathbb{N} \), let \( L_n = \{ (1/n, y) : 0 \leq y \leq 1 \} \) and let \( L_0 = \{ (0, y) : 0 < y < 1 \} \). Set \( X = P \setminus \bigcup_{n=0}^{\infty} L_n \). Then \( P \) is a Peano compactification of \( X \); however, \( X \) is not S-metrizable.

Suppose \( \rho \) is an S-metric for \( X \) and let \( A = \{ (x, 1) : 0 \leq x \leq 1 \} \) and \( B = \{ (x, 0) : 0 \leq x \leq 1 \} \). Then \( A \) and \( B \) are compact and hence \( \rho(A, B) = \varepsilon > 0 \). Now the components \( C_1, C_2, \ldots \) of \( X \setminus (A \cup B) \) have limit points in each of \( A \) and \( B \). Thus, any collection of connected sets of \( \rho \)-diameter less than \( \varepsilon/3 \) that covers a component \( C_n \) has at least one such connected subset lying entirely in \( C_n \). This implies that \( \rho \) is not an S-metric for \( X \); however, if \( d \) is the relative metric on \( X \) inherited from the usual metric on \( P \), \( d \) is almost convex and totally bounded.

3. **RELATED RESULTS.**

A compatible normal sequence in a space \( Z \) is a sequence \( U_1, U_2, \ldots \) of open covers of \( Z \) such that \( U_{n+1} \) star-refines \( U_n \) for \( n = 1, 2, \ldots \) and so, for any \( x \in Z \), \( \{ St(x, U_n) : n = 1, 2, \ldots \} \) is a neighborhood base for \( x \) [5].

**THEOREM 3.1.** [6, Prop. 23.4] A \( T_0 \)-space is metrizable if and only if it has a compatible normal sequence.

**COROLLARY 3.1.** A metric space \( X \) is totally bounded if and only if \( X \) has a compatible normal sequence \( U_1, U_2, \ldots \) where each \( U_n \) is a finite cover of \( X \).

**PROOF.** Suppose \((X, d)\) is totally bounded. It follows from the total boundedness of \((X, d)\) that there is a finite open cover \( U_1 \) of \( X \) such that \( \delta_d(U) = 1/3 \) for all \( U \in U_1 \), where \( \delta_d(U) = \sup\{d(x, y) : x, y \in U\} \), the \( d \)-diameter of \( U \). Since \( U_1 \) is finite, there is a Lebesgue number \( \varepsilon_1 < 3^{-2} \) such that, if \( d(x, y) < \varepsilon_1 \), then \( x \) and \( y \) be in some member of \( U_1 \). Again, by the total boundedness of \((X, d)\), there is a finite open cover \( V_1 \) of \( x \) such that \( \delta_d(V) < \varepsilon_1 \). If \( \varepsilon_2 < \varepsilon_1 \) is a Lebesgue number for \( V_1 \) and \( U_2 \) is any fini
open cover of X such that $\delta_{d}(U) < \varepsilon_2$ for any $U \subseteq U_2$, then $U_2$ star-refines $U_1$. Continue in this manner and obtain a compatible normal sequence $U_1, U_2, \ldots$ for X.

On the other hand, suppose $U_1, U_2, \ldots$ is a compatible normal sequence for X where each $U_n$ is finite. Then, in the usual metric $\rho$ for X that is associated with $U_1, U_2, \ldots$ as given by S. Willard [6], $\delta_{\rho}(U) < 2^{n-1}$ and $U \subseteq U_n$, $n = 2, 3, \ldots$. It then follows that, since each $U_n$ is finite, $\rho$ is a totally bounded metric for X. This completes the proof.

**Corollary 3.2.** A metric space $(X, d)$ is $S$-metrizable if and only if it has a compatible normal sequence $U_1, U_2, \ldots$ where each $U_n$ is a finite cover and the members of $U_n$ are connected sets.

**Proof.** The necessity follows from the argument above, together with the observation that the covers $U_1, U_2, \ldots$ can be selected so as to consist of finitely many open and connected sets.

The sufficiency. We observe that, if $U_1, U_2, \ldots$ is a compatible normal sequence for X where each $U_n$ is finite and the members of $U_n$ are connected sets and if $\rho$ is the usual metric associated with $U_1, U_2, \ldots$ as given in [6], then, for $U \subseteq U_n$, $\delta_{\rho}(U) < 2^{n-1}$, $n = 2, 3, \ldots$ and the sets $U \subseteq U_n$ are connected. Thus, for any $\varepsilon > 0$ and $k \in \mathbb{N}$ so that $0 < 2^{-k} < \varepsilon$, $x = \cup \{U : U \subseteq U_k \}$ is a finite cover of X by connected sets of $\rho$-diameter less that $\varepsilon$. This completes the proof.

**Theorem 3.2** [2]. A connected metric space X has an almost convex metric if and only if it has a compatible normal sequence $U_1, U_2, \ldots$ such that (i) each pair of points that is covered by either an element of $U_{n+1}$ or the union of a pair of intersecting elements of $U_{n+1}$ can be covered by an element of $U_n$ and (ii) each pair of points that can be covered by an element of $U_n$ can be covered by the union of two intersecting elements of $U_{n+1}$.

It is, apparently, very difficult to combine the total boundedness (finiteness) conditions of Corollaries 3.1 and 3.2 and the intersection-type properties of Theorem 3.2. It would be very desirable to do so in light of the results of the previous section.
REFERENCES

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