FUNCTIONS IN THE SPACE $R^2(E)$ AT BOUNDARY POINTS OF THE INTERIOR

EDWIN WOLF

Department of Mathematics
Marshall University
Huntington, West Virginia 25701

(Received September 15, 1982)

ABSTRACT. Let E be a compact subset of the complex plane $\mathbb{C}$. We denote by $R(E)$ the algebra consisting of (the restrictions to E of) rational functions with poles off E . Let m denote 2 - dimensional Lebesgue measure. For $p \geq 1$, let $R^p(E)$ be the closure of $R(E)$ in $L^p(E, dm)$.

In this paper we consider the case $p = 2$. Let $x \in \partial E$ be a bounded point evaluation for $R^2(E)$. Suppose there is a $C > 0$ such that x is a limit point of the set $S = \{y | y \in \text{Int } E, \text{Dist}(y, \partial E) \geq C|y - x|\}$. For those $y \in S$ sufficiently near x we prove statements about $|f(y) - f(x)|$ for all $f \in R(E)$.

KEY WORDS AND PHRASES. Rational functions, compact set L^p - spaces, bounded point evaluation, admissible function.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. 30A98, 46E99.

1. INTRODUCTION AND DEFINITIONS

Let E be a compact subset of the complex plane $\mathbb{C}$. We denote by $R(E)$ the algebra consisting of (the restrictions to E of) rational functions with poles off E . Let m denote 2 - dimensional Lebesgue measure. For $p \geq 1$, let $R^p(E)$ be the closure of $R(E)$ in $L^p(E, dm)$. A point $x \in E$ is said to be a bounded point evaluation (BPE) for $R^p(E)$ if there is a constant F such that

$$|f(x)| \leq F \cdot \left\{ \int_E |f(z)|^p dm(z) \right\}^{\frac{1}{p}} \text{ for all } f \in R(E).$$

In [4] we studied the smoothness properties of functions in $R^p(E)$, $p > 2$, at BPE's. When $p = 2$, the situation is quite different (see Fernström and Polking [2] and Fernström [1]). In [5] we showed that at certain BPE's the functions in $R^2(E)$ have the following smoothness property: Let $x \in \partial E$ be both a BPE for $R^2(E)$ and the vertex of a sector contained in $\text{Int } E$. Let L be a line segment that bisects the sector and has an end point at x . Then for each $\epsilon > 0$ there is a $\delta > 0$ such that if $y \in L$ and $|y - x| < \delta$, $|f(y) - f(x)| \leq \epsilon \|f\|_2$ for all $f \in R(E)$. The goal of this paper is to extend this result to certain cases where there may not be a sector in $\text{Int } E$ having vertex at x , but x is still a limit point of $\text{Int } E$.

If $x \in E$ is a BPE for $R^2(E)$, there is a function $g \in L^2(E)$ such that $f(x) = \int_E fg \, dm$ for any $f \in R(E)$. Such a function g is called a representing function for x .

A point $x \in E$ is a bounded point derivation (BPD) of order s for $R^2(E)$ if the map $f \mapsto f^{(s)}(x)$, $f \in R(E)$, extends from $R(E)$ to a bounded linear functional on $R^2(E)$.

Let $A_n(x)$ denote the annulus $\{z | 2^{-n-1} \leq |z - x| \leq 2^{-n}\}$. Let $A'_n(x) = \{z | 2^{-n-2} \leq |z - x| \leq 2^{-n+1}\}$. If $x = 0$, we will denote $A_n(0)$ by A_n and $A'_n(0)$ by A'_n .

For an arbitrary set $X \subset \mathbb{C}$ we let $C_2(X)$ denote the Bessel capacity of X which is defined using the Bessel kernel of order 1 (see [3]).

We say that ϕ is an admissible function if ϕ is a positive, non-decreasing function defined on $(0, \infty)$, and $r \cdot \phi(r)^{-1}$ is nondecreasing and tends to zero when $r \rightarrow 0^+$.

Using the techniques of [4] and [2] one can prove:

THEOREM 1.1. Let s be a nonnegative integer and E a compact set. Suppose that x is a BPE for $R^2(E)$ and ϕ is admissible. Then x is

represented by a function $g \in L^2(E)$ such that

$$\frac{g}{(z-x)^s \cdot \phi(|z-x|)} \in L^2(E)$$

if and only if $\sum_{n=0}^{\infty} 2^{2n(s+1)} \phi(2^{-n})^{-2} C_2(A_n(x) - E) < \infty$.

2. THE MAIN RESULTS

Let $x \in \partial E$ be a BPE for $R^2(E)$. We may assume that $x = 0$ and that $E \subset \{|z| < 1\}$. Suppose there is a positive constant C such that 0 is a limit point of the set $S = \{y | y \in \text{Int } E, \text{Dist}(y, \partial E) \geq C|y|\}$. We will construct a function $g \in L^2(E)$ which represents 0 for $R^2(E)$ and has support disjoint from S .

LEMMA 2.1. Let $0 \in \partial E$ be a BPE for $R^2(E)$. Suppose there is a positive constant C such that 0 is a limit point of the set $S = \{y | y \in \text{Int } E, \text{Dist}(y, \partial E) \geq C|y|\}$. Then there is a function $g \in L^2(E)$ such that:

- (i) g represents 0 for $R^2(E)$,
- (ii) $m((\text{supp } g) \cap S) = 0$,

$$k = \left[\frac{n-2}{2} \right] + 1$$

- (iii) For all $n \geq 2$, $\int_{A_n \cap E} |g|^2 dm \leq F \sum 2^{2k} C_2(A'_{2k+1} - E)$

$$k = \left[\frac{n-2}{2} \right]$$

where F is a constant independent of n .

PROOF. For each i , $i = 0, 1, 2, \dots$ consider all the intersections of the set $A_i = \{z | 2^{-i-1} \leq |z| \leq 2^{-i}\}$ with the bounded components of $C - E$. Let Y_i be the closure of the union of these intersections. Since Y_i is compact, it can be covered by finitely many open discs of radius $< C3^{-1}2^{-i-1}$. Let the union (finite) of these discs be denoted by B_i . The set B_i is bounded by finitely many closed Jordan curves each of which is the union of finitely many circular arcs. Each set B_i is contained in a set C_i bounded by finitely many closed Jordan curves Γ_{ij} , $j = 1, 2, \dots, n_i$ such that if z belongs to any one of these

curves, $\text{Dist}(z, B_i) = 2C3^{-1}2^{-i-1}$.

Now for each k , $k = 0, 1, 2, \dots$ choose a function $\lambda_k \in C_0^1$

such that:

$$(1) \quad \text{supp } \lambda_k \subset A_{2k+1}'$$

$$(2) \quad \lambda_k(z) = 1 \text{ for } z \in \{z \mid 2^{-2k-2} \leq |z| \leq 2^{-2k-1}\} \cap B_{2k+1}$$

$$(3) \quad \lambda_k(z) = 0 \text{ for } z \notin \bigcup_{i=0}^{\infty} C_i$$

$$(4) \quad \left| \frac{\partial \lambda_k(z)}{\partial x_1} \right| \leq F_1 \cdot 2^{2k+1}, \quad \left| \frac{\partial \lambda_k(z)}{\partial x_2} \right| \leq F_2 \cdot 2^{2k+1}$$

where $z = x_1 + ix_2$ and F_1 and F_2 are constants independent of k .

$$(5) \quad \lambda_k(z) + \lambda_{k+1}(z) = 1 \text{ for } z \in \{z \mid 2^{-2k-3} \leq |z| \leq 2^{-2k-2}\} \cap B_{2k+2}.$$

Given any $\epsilon > 0$ we use a lemma of Fernström and Polking [2] to obtain functions $\psi_k \in C^\infty$ such that:

$$(1) \quad \psi_k(z) \equiv 1 \text{ for } z \text{ near } A_k' - \{z \mid \text{Dist}(z, E) < \epsilon\}.$$

$$(2) \quad \int_{|z| \leq 2^{-k+1}} |D^\beta \psi_k(z)|^2 dm(z) \leq F \cdot 2^{-2k(1-|\beta|)} C_2(A_k' - E)$$

for $\beta = (0, 0)$, $(0, 1)$, and $(1, 0)$. Here the constant F is independent of ϵ and k .

Since $\text{supp } \lambda_k \subset A_{2k+1}'$, we have $\psi_{2k+1} \cdot \lambda_k = \lambda_k$ on the set $\{z \mid \text{Dist}(z, E) \geq \epsilon\}$. Thus $\sum_0^\infty \psi_{2k+1} \cdot \lambda_k \equiv 1$ on $\{|z| \leq 4^{-1}\} - \{z \mid \text{Dist}(z, E) < \epsilon\}$. Choose $\chi \in C_0^\infty$ with $\chi(z) \equiv 1$ near E . Set $h(z) = \chi(z) \cdot \frac{1}{\pi z}$. For each double index $\beta = (0, 0)$, $(0, 1)$, and $(1, 0)$ there is a constant F_β such that

$$|D^\beta h(z)| \leq F_\beta \cdot |z|^{-1-|\beta|}.$$

$$\text{Set } f_\epsilon = h \cdot \sum_0^\infty \psi_{2k+1} \cdot \lambda_k = \sum_0^\infty \psi_{2k+1} \cdot h_k$$

where $h_k = \lambda_k h$.

Since $\text{supp } \lambda_k \subset A_{2k+1}'$, the above inequalities imply that

$$|D^\beta h_k(z)| \leq F_\beta 2^{(2k+1)(1+|\beta|)}$$

The subadditivity of C_2 and the convergence of $\sum_{k=0}^{\infty} 2^{2k} C_2(A_k - E)$ (see Theorem 1.1) imply that the net $\{f_\epsilon\}$ is bounded in L_1^2 . There is a subsequence that converges weakly to a function $f \in L_{1,loc}^2$

which satisfies $f(z) = \frac{\chi(z)}{\pi z}$ for $z \in \mathbb{C} - E$ and $f(z) = 0$ for every

$z \in E \cap \{z | \text{Dist}(z, \partial E) \geq C|z|\}$. Set $g = -\frac{\partial}{\partial \bar{z}} f$. Then $g \in L^2(E)$

since $f \in L_1^2(E)$, and g is a representing function for 0. The proof of (iii) proceeds as in [5].

The above lemma can be used to prove the following theorem in almost the same way that in [5] Lemma 5.1 is used to prove Theorem 5.1.

THEOREM 2.1. Let $0 \in \partial E$ be a BPE for $R^2(E)$. Let C be a positive constant such that 0 is a limit point of the set

$S = \{y | y \in \text{Int } E, \text{Dist}(y, \partial E) \geq C|y|\}$. Let g be a representing function for 0 and suppose that $g(z) \cdot \phi(|z|)^{-1} \in L^2(E)$ where ϕ is an admissible function. Then for any $\epsilon > 0$ there is a $\delta > 0$ such that if $y \in S$ and $|y| < \delta$,

$$|f(y) - f(0)| \leq \epsilon \phi(|y|) \|f\|_2$$

for all $f \in R(E)$.

Using this theorem and the methods in [4] one can prove:

COROLLARY 2.1. Suppose that all the conditions of Theorem 2.1 hold. Suppose, moreover, that s is a positive integer such that $g(z) \cdot z^{-s} \cdot \phi(|z|)^{-1} \in L^2(E)$. Then for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $y \in S$ and $|y| < \delta$,

$$|f(y) - f(0) - \frac{f'(0)}{1!} (y - 0) - \dots - \frac{f^{(s)}(0)}{s!} (y - 0)^s| \leq \epsilon |y - 0|^s \phi(|y|) \|f\|_2$$

for all $f \in R(E)$.

Finally, there is a corollary with weaker preconditions.

COROLLARY 2.2. Let $0, g$, and ϕ be as in Theorem 2.1. Suppose there is a positive constant C such that 0 is a limit point of the set

$$S = \{y \mid y \in \text{Int } E, \text{Dist}(y, \partial E) \geq C \cdot \phi(|y|)|y|\}$$

Then for each $\epsilon > 0$ there is a $\delta > 0$ such that if $y \in S$ and $|y| < \delta$,

$$|f(y) - f(0)| \leq \epsilon \|f\|_2 \text{ for all } f \in R(E).$$

The proof is similar to the proof of Theorem 2.1.

One uses the fact that there exists an admissible function $\bar{\phi}$ such that $g \cdot \bar{\phi}^{-1} \cdot \bar{\phi}^{-1} \in L^2(E)$.

3. EXAMPLES

EXAMPLE 1. We will construct a compact set E such that $0 \in \partial E$, 0 is a BPE for $R^2(E)$, and 0 is a limit point of $\text{Int } E$. Let $D = \{|z| \leq 1\}$. Let D_i , $i = 1, 2, 3, \dots$, be the open disc centered on the positive real axis at $3 \cdot 2^{-i-3}$ and having radius $r_i = \exp(-2^{2i} i^2)$.

Let $E = D - \bigcup_{i=1}^{\infty} D_i$. Then since $C_2(B(x, r)) \leq F(\log \frac{1}{r})^{-1}$, $r \leq r_0 < 1$, (see [3]), we have

$$\sum_{n=1}^{\infty} 2^{2n} C_2(A_n - E) = \sum_{n=1}^{\infty} 2^{2n} C(D_n) \leq F \cdot \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Thus 0 is a BPE for $R^2(E)$. If C is a positive constant sufficiently small (any positive number $< \frac{1}{2}$ will do), the set $\{y \mid y \in \text{Int } E, \text{Dist}(y, \partial E) \geq C|y|\}$ intersects the positive real axis in a sequence of disjoint intervals $[a_n, b_n]$ such that $b_n \rightarrow 0$.

EXAMPLE 2. Next we construct a compact set E which is like Example 1 in that 0 is a limit point of $\text{Int } E$ and a BPE for $R^2(E)$. In this example, however, there exists no sequence $\{y_n\} \subset \text{Int } E$ such that $|f(y_n) - f(0)| \leq \epsilon \|f\|_2$ for all $f \in R(E)$ if $|y_n| < \delta$. We will use

important parts of Fernström's construction in [1]. Let F be a positive constant such that $C_2(B(z, r)) \leq F(\log \frac{1}{r})^{-1}$ for all $r, r \leq r_0 < 1$. Choose $\alpha, \alpha \geq 1$ such that

$$\frac{F}{\alpha} \cdot \sum_{n=1}^{\infty} \frac{1}{n \log^2 n} < C_2(B(0, 1/2)).$$

Let A_0 be the closed unit square with center at 0. Cover A_0 with 4^n squares of side 2^{-n} . Call the squares $A_n^{(i)}, i = 1, 2, \dots, 4^n$. In every set $A_n^{(i)}$ put an open disc $B_n^{(i)}$ such that $B_n^{(i)}$ and $A_n^{(i)}$ have the same center, and the radius of $B_n^{(i)}$ is $\exp(-\alpha 4^n n \log^2 n)$. Let $D_i, i = 1, 2, 3, \dots$ be an open disc centered on the positive real axis such that $D_i \subset \{z | 2^{-i-1} \leq |z| \leq 2^{-i}\}$ and $r_i = \exp(-2^{2i} i^2)$. For each $n, n = 1, 2, 3, \dots$, let $G_n = \bigcup_i B_n^{(i)}$ where the summation is over those indices i such that $1 \leq i \leq 4^n$ and $B_n^{(i)} \cap (\bigcup_1^{\infty} D_i) = \varnothing$. Set $E_1 = A_0 - \bigcup_{n=2}^{\infty} G_n$. Then $R^2(E_1)$ has no BPE's in ∂E_1 as is shown in [1].

Now replace a suitable number of the discs

$B_n^{(i)}, B_n^{(i)} \subset \bigcup_{j=2}^{\infty} G_j$, to obtain a compact set E_2 such that 0 is the only boundary point of E_2 that is a BPE for $R^2(E_2)$, (see [1]).

This can be done so that $\text{Int } E_2 = \bigcup_1^{\infty} D_i$. If $y \in \text{Int } E_2$, let $\text{norm}(y)$ denote the norm of "evaluation at y " as a linear functional on $R^2(E_2)$. Then if $\{y_k\} \subset D_i$, and $y_k \rightarrow \partial D_i$, $\text{norm}(y_k) \rightarrow \infty$; otherwise some point on ∂D_i would be a BPE for $R^2(E_2)$.

For each i choose an open disc $D_i' \subset D_i$ such that D_i' and D_i are concentric and such that if $y \in D_i - D_i'$, then $\text{norm}(y)$ for the space $R^2(E_2 - D_i')$ is greater than i .

Now let $E = E_2 - \bigcup_{i=1}^{\infty} D_i$.

The radii of the D_i^1 are so small that 0 is also a BPE for $R^2(E)$. Let $\{y_n\}$ be any sequence in $\text{Int } E$ such that $y_n \rightarrow 0$. Let $\text{norm}(y_n) = \text{norm of "evaluation at } y_n \text{" on } R^2(E)$. Then for no $\epsilon > 0$ is there a $\delta > 0$ such that if $|y_n| < \delta$, $|f(y_n) - f(0)| \leq \epsilon \|f\|_2$ for all $f \in R(E)$.

EXAMPLE 3. Let ϕ be an admissible function. Obtain a compact set E in the same way that the set E_2 was obtained in Example 2 so that:

- (1) D_i is centered at $3 \cdot 2^{-i-2}$ and has radius $r_i = \phi(3 \cdot 2^{-i-2}) \cdot 2^{-i-2}$
- (2) $\sum_{n=0}^{\infty} 2^{2n} \cdot \phi(2^{-n})^{-2} C_2(A_n(0) - E) < \infty$, and
- (3) $\sum_{n=0}^{\infty} 2^{2n} C_2(A_n(x) - E) = \infty$ for $x \neq 0$, $x \notin \bigcup D_i$.

Let $y_i = 3 \cdot 2^{-i-2}$. Then by the choice of r_i , $\text{Dist}(y_i, E) \geq 3^{-1} \cdot \phi(|y_i|) |y_i|$. But there is no $C > 0$ such that $\text{Dist}(y_i, \partial E) \geq C |y_i|$ for all i . Hence Corollary 2.2 applies to the sequence $\{y_i\}$ but Theorem 2.2 does not.

REFERENCES

1. Fernström, C., Some remarks on the space $R^2(E)$, Math. Reports, University of Stockholm 1982.
2. Fernström, C. and Polking, J. C., Bounded point evaluations and approximation in L^p by solutions of elliptic partial differential equations. J. Functional Analysis, 28, 1-20 (1978).
3. Meyers, N. G., A theory of capacities for potentials of functions in Lebesgue classes, Math. Scand., 26 (1970), 255-292.
4. Wolf, E., Bounded point evaluations and smoothness properties of functions in $R^p(X)$, Trans. Amer. Math. Soc. 238 (1978), 71-88.
5. Wolf, E., Smoothness properties of functions in $R^2(X)$ at certain boundary points, Internat. J. Math. and Math. Sci., 2 (1979), 415-426.